Self-similarity, Multifractionality and Multifractality

A thesis submitted for the Degree of
Master of Philosophy

Australian National University

Allan Sly

March 1, 2006

Declaration

The work in this thesis is my own except where otherwise stated.

Allan Sly

Allan Sly

Acknowledgements

This thesis would not have been possible without the assistance of many people. Once again my greatest thanks must go to my supervisor Chris Heyde for all his guidance over the last year which continued unfailingly during his several trips to the other side of the world. His advice on problems and comments on numerous drafts have helped shape my work and this thesis.

I was very fortunate to have worked with a group of other stochastic analysis students Ben, Damien and Priya with whom I spent much time both mathematically and socially. Ben and Priya also made useful comments on drafts of this thesis. It has been a pleasure to have shared an office with Ben, Damien, Griff, Kester, Pourus and Stéphanie and to have spent time with the other graduate and honours students. I would like to thank all the people in the Mathematical Sciences Institute who have given me the excellent education that I have received and the support which has made this year successful.

Most important of all has been the love and support of my parents this year.

Abstract

This thesis examines three separate problems involving stochastic processes which exhibit scaling behavior. We begin the thesis with a review of self-similar stochastic processes, their importance in modelling and some important results.

Multifractional Brownian motion and its variants are a class of stochastic processes designed to model varying roughness which is measured in terms of the Hölder exponent. It has local asymptotic scaling. We show that the construction of multifractional Brownian motion has the unfortunate consequence that changes in roughness lead to large changes in magnitude. We present an alternative called integrated fractional white noise which retains the local scaling and roughness without the large swings in magnitude. We then present an estimator for the local Hölder exponent.

The next chapter looks at the complex scaling exhibited by financial returns and compares two subordinator models one of which is self-similar while the other is multifractal. We find that the evidence for the multifractal model can be explained by the heavy tails of the process. The scaling found in the self-similar model, however, is significantly different from trivial independent scaling.

The final chapter looks at the interaction of heavy tails and long range dependence. We extend the results of [Dobrushin and Major, 1979] and [Taqqu, 1979] to functionals of long range dependent Gaussian sequences with infinite variance and finite mean. This result is used to prove a limit theorem for a process from [Heyde and Leonenko, 2005].

Contents

1 Introduction **10**

1.1 Introduction to Self-similar Processes 10

1.2 Summary of Thesis 14

2 Integrated Fractional White Noise **17**

2.1 Introduction 17

2.2 Variance Estimate for Integrated Fractional White Noise 20

2.3 Local Self-similarity and Path Properties 25

2.4 Identification of $H(t)$ 28

2.5 Generalizations 36

3 Multifractal and Monofractal Subordinator models in Finance **37**

3.1 Multifractal Stochastic Processes 39

3.2 Brownian motion in Multifractal Time 41

3.3 Evidence for Multifractality 42

3.4 FATGBM 50

3.4.1 Definition 50

3.4.2 Activity Times for General ν 52

3.4.3 Concentration of Extreme Values 53

3.5 Conclusion 54

4 Noncentral Limit Theorem for Infinite Variance Functionals **55**

4.1 Introduction 55

4.2 Proof 57

4.3 Applications 67

Bibliography **69**

Chapter 1

Introduction

Invariance under scaling is a fractal property found in a wide range of natural phenomena. For stochastic processes invariance under scaling means invariance, or asymptotic invariance, in distribution under appropriate scaling of time and space. Processes exhibiting scaling are the natural analogues of fractals in stochastic processes. Complex scaling behavior of stochastic processes exists in a diverse range of fields including physics, climatology, geology, finance and even internet data traffic. There is a wide literature which outlines the importance of scaling including [Mandelbrot, 1977], [Mandelbrot, 1997] and [Embrechts and Maejima, 2002]. This chapter gives a brief overview of the important results and ideas in self-similar processes and the associated concept of long range dependence. Section 2 of this chapter outlines the new results in this thesis which relate to various types of scaling in stochastic processes and their consequences.

1.1 Introduction to Self-similar Processes

The most important class of scaling stochastic processes are the self-similar stochastic processes.

1.1.1 Definition (Self-similar Stochastic Process). A stochastic process $X(t)$ is self-similar if for each $a > 0$ there exists some b such that

$$\{Y(at)\} \stackrel{d}{=} \{bY(t)\}$$

where equality is in finite dimensional distributions.

This property means that when both time and space are scaled appropriately the whole process is unchanged in finite dimensional distributions. Making the very reasonable assumption of stochastic continuity at $t = 0$ the scaling can be shown to obey a power law.

1.1.2 Theorem ([Lamperti, 1962]). If $Y(t)$ is self-similar and stochastically continuous at $t = 0$ then there exists a unique $H > 0$ such that for all $a > 0$

$$\{Y(at)\} \stackrel{d}{=} \{a^H Y(t)\}$$

where equality is in finite dimensional distributions.

A self-similar process with parameter H is denoted H-ss and denoted H-sssi if it also has stationary increments. The parameter H is known as the Hurst parameter named for the British engineer Harold Hurst whose work on Nile river data played an important role in the development of self-similar processes. An H-sssi process with finite mean must have $0 < H \leq 1$ and $X(0) = 0$ a.s. If it has finite mean and $H = 1$ then it is the degenerate process $X(t) = tX$ for some random variable X. If it has finite mean and $0 < H < 1$ then $EX(t) = 0$

The most important and most basic self-similar processes are derived from the stable laws. Brownian motion is $\frac{1}{2}$-sssi since it is easily verified that $B(at) \stackrel{d}{=} a^{\frac{1}{2}} B(t)$ where equality is in finite dimensional distributions.

Suppose X_i is a sequence of iid random variables in the domain of attraction of some stable law with index of stability α. Then by the Central Limit Theorem there are sequences $\{a_n\}$ and $\{b_n(> 0)\}$ of constants such that

$$b_n^{-1} \sum_{i=1}^{n} X_i - a_n \stackrel{d}{\to} Z$$

and

$$b_n^{-1} \sum_{i=1}^{\lfloor nt \rfloor} X_i - a_n \stackrel{d}{\to} Z(t)$$

in finite dimensional distributions where $Z(t)$ is α-stable Lévy motion which is $\frac{1}{\alpha}$-sssi stable. For more details on stable processes see [Samorodnitsky and Taqqu, 1994]. While all these processes have independent increments they are in fact the special case. All other H-sssi stochastic processes have dependent increments.

When a process is H-sssi and has finite variance its covariance structure is completely determined, up to a constant, since it follows that

$$\begin{aligned} EY(t)Y(s) &= \frac{1}{2}[E[Y(s)^2] + E[Y(t)^2] - E[(Y(t) - Y(s))^2]] \\ &= \frac{1}{2}[E[Y(s)^2] + E[Y(t)^2] - E[(Y(t - s))^2]] \\ &= \frac{1}{2}[s^{2H} + t^{2H} - |t - s|^{2H}]E[Y(1)^2]. \end{aligned} \tag{1.1}$$

Since the mean-covariance structure completely determines the finite dimensional distributions of a Gaussian process there is only one Gaussian H-sssi process for each H. Fractional Brownian motion was introduced in [Mandelbrot and Van Ness, 1968] as a moving average stochastic integral.

1.1.3 Definition (Fractional Brownian Motion). Fractional Brownian motion is given by the Itô integral

$$B_H(t) = \frac{1}{\Gamma(H + \frac{1}{2})} \int_{\mathbb{R}} \varphi_H(t - x) - \varphi_H(-x) dB(x) \tag{1.2}$$

where $B(x)$ is Brownian motion and

$$\varphi_H(x) = \begin{cases} x^{H - \frac{1}{2}} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Fractional Brownian motion is H-sssi and we will denote it by $B_H(t)$. When $H = \frac{1}{2}$ this simplifies to standard Brownian motion. When we replace the Wiener measure with an α-stable measure and $H - \frac{1}{2}$ with $H - \frac{1}{\alpha}$ we get fractional stable motion, a self-similar process with stable increments. Another representation of fractional Brownian motion is the harmonizable representation which is given by

$$M_H(t) = \mathrm{Re} \int_{\mathbb{R}} \left(\frac{e^{i\xi t} - 1}{|\xi|^{H+1/2}} \right) \widetilde{W}(d\xi)$$

where $\widetilde{W} = W_1 + iW_2$ and W_1 and W_2 are independent Wiener measures.

An important concept which is closely linked to self-similarity is long range dependence.

1.1.4 Definition. A stationary sequence with finite variance X_n is long range dependent if

$$\sum_{n=0}^{\infty} EX_0X_n = \infty. \tag{1.3}$$

The concept of long range dependence can be extended to random variables with infinite variance but there are a number of competing definitions (see [Heyde and Yang, 1997]).

When $Y(t)$ is a finite variance H-sssi process with $\frac{1}{2} < H < 1$ and $X_n = Y(n) - Y(n-1)$ it follows from equation (1.1) that X_n is a long range dependent sequence.

Since self-similarity is defined in terms of finite dimensional distributions most statistical tests instead look for long range dependence. Long range dependence is more easily identified as it is simply a property of the covariance structure. It is common to assume a self-similar model when long range dependence is found. The limit theorems discussed later in this introduction show that a broad range of long range dependent sequences are asymptotically self-similar which justifies this assumption.

The following result shows that any asymptotic limit result must be of the form of a self-similar process.

1.1.5 Theorem. ([Lamperti, 1962]) Suppose that X_n is a stationary sequence and that b_n is a sequence of real numbers such that $b_n \to \infty$ as $n \to \infty$. Then if

$$\frac{1}{b_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i \xrightarrow{d} Y_t \tag{1.4}$$

as $n \to \infty$ in finite dimensional distributions there exists an H such that Y_t is H-sssi. Furthermore $b_n = L(n)n^H$ where $L(n)$ is a slowly varying function at infinity (that is for $c > 0$, $\lim_{n\to\infty} L(cn)/L(n) = 1$).

For short range dependent sequences the limit will typically be Brownian motion or an α-stable Lévy motion. There are a number of important limit theorems for long range dependent random variables. The Gaussian case is very simple (see for example [Embrechts and Maejima, 2002]).

1.1.6 Theorem. Let X_i be a stationary sequence of Gaussian random variables such that

$X_i \stackrel{d}{=} N(0,1)$ and $EX_0X_n \sim n^{2H-2}$ where $\frac{1}{2} < H < 1$. Then,

$$n^{-H}\sum_{i=1}^{\lfloor nt \rfloor} X_i \xrightarrow{d} B_H(t)$$

as $n \to \infty$ in finite dimensional distributions where $B_H(t)$ is fractional Brownian motion.

Next Taqqu extended a result of Rosenblatt as follows.

1.1.7 Theorem. ([Taqqu, 1975]) Let X_i be a stationary sequence of Gaussian random variables such that $X_i \stackrel{d}{=} N(0,1)$ and $EX_0X_n \sim n^{2H-2}$ where $\frac{3}{4} < H < 1$. Then,

$$n^{-(1-2(1-H))}\sum_{i=1}^{\lfloor nt \rfloor} (X_i^2 - 1) \xrightarrow{d} Z_H(t)$$

as $n \to \infty$ in finite dimensional distributions where $Z_H(t)$ is the Rosenblatt process.

Fractional Brownian motion and the Rosenblatt distribution are just the first two of a series of processes. In fact a similar limit theorem can be proved for any Borel function f such that $f(X_i)$ has finite variance.

The Hermite polynomials are defined $h_k(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^k}{dx^k} e^{\frac{x^2}{2}}$ so $h_0(x) = 1$, $h_1(x) = x$, $h_2(x) = x^2 - 1$. When $X, Y \sim N(0,1)$ are jointly Gaussian

$$Eh_k(X)h_j(Y) = \begin{cases} k!(EXY)^k & k = j, \\ 0 & k \neq j. \end{cases}$$

The Hermite polynomials form a basis for $L^2(\mathbb{R}, e^{-\frac{x^2}{2}})$ so when f is a function such that $Ef(X)^2 < \infty$,

$$f(X) = \sum_{k=0}^{\infty} f_k h_k(X)$$

where the sum is interpreted by convergence in L^2 and $f_k = \frac{1}{k!}E(f(X)h_k(X))$. The Hermite rank of f is the smallest k such that $f_i \neq 0$. For more details on Hermite polynomials and their relationship to Gaussian Hilbert spaces see [Holden et al., 1996] or [Janson, 1997].

1.1.8 Theorem. ([Taqqu, 1979], [Dobrushin and Major, 1979]) Let X_i be a stationary sequence of Gaussian random variables such that $X_i \stackrel{d}{=} N(0,1)$ and $EX_0X_n \sim n^{2H-2}$. Let f be a function such that $Ef(X_1) = 0$, $Ef(X_1)^2 < \infty$. Let $\kappa > 0$ be the Hermite rank of f. Then if $1 - \kappa(1-H) > \frac{1}{2}$

$$n^{-(1-\kappa(1-H))}\sum_{i=1}^{\lfloor nt \rfloor} f(X_i) \xrightarrow{d} f_\kappa (H(2H-2))^{-1} R_{\kappa, 1-\kappa(1-H)}$$

as $n \to \infty$ where $R_{\kappa,1-\kappa(1-H)}$ is the κ-Hermite process with $(1-\kappa(1-H))$-self-similarity. But if $1 - \kappa(1-H) < \frac{1}{2}$,

$$n^{-\frac{1}{2}}\sum_{i=1}^{\lfloor nt \rfloor} f(X_i) \xrightarrow{d} CB(t)$$

where $B(t)$ is Brownian motion.

The first Hermite process is fractional Brownian motion while the second is the Rosenblatt process. They have the multiple Itô integral representation

$$R_{k,H'}(t) \stackrel{d}{=} C^k \int_{\mathbb{R}^k} \int_0^t \prod_{i=1}^k |s - x_i|^{H_0 - \frac{3}{2}} ds dB(x_1) \dots dB(x_k)$$

where $H_0 = 1 - \frac{1-H'}{k}$ and

$$C = \left(2\Gamma \left(H_0 - \frac{1}{2} \right) \cos \left(\frac{\pi}{2} \left(H_0 - \frac{1}{2} \right) \right) \right)^{-1}.$$

This integral representation does not provide much information about the distribution of the process about which very little is known.

Up until now our discussion has focussed on the long range properties of self-similar processes. However, the small scale properties of self-similar processes are also interesting as their sample paths can behave as rough fractal functions. We measure roughness in terms of the Hölder exponent.

Recall that for $0 < \gamma \leq 1$ a function is γ-Hölder continuous if for some constant C and for all x and y, $|f(x) - f(y)| \leq C|x - y|^\gamma$. When $\gamma = 1$ this is Lipschitz continuity. The notion of the Hölder exponent generalizes Hölder continuity to a point.

1.1.9 Definition (Hölder Exponent)**.** A continuous function $g(x)$ has Hölder exponent H at x_0 if

$$\sup\{\gamma : \lim_{h \to 0} |h|^{-\gamma} |g(x_0 + h) - g(x_0)| = 0\} = H.$$

A process which is H-Hölder continuous has Hölder exponent greater than or equal to H.

1.1.10 Theorem (Hölder Exponent, [Samorodnitsky and Taqqu, 1994])**.** Let $Y(t)$ be H-sssi and have moments of all orders. With probability 1, $Y(t)$ has continuous paths with Hölder exponent H.

This result covers fractional Brownian motion and all the Hermite processes. An important application for this is in terrain simulation where the terrain is known to have a particular roughness.

For more detailed review of self-similar processes see [Embrechts and Maejima, 2002]. While self-similar scaling is the most studied type of scaling for stochastic processes other variants include multifractal scaling and local asymptotic self-similar scaling. These types of scaling are concerned with the small time scale behavior rather than long range properties of the processes.

1.2 Summary of Thesis

Chapters 2, 3 and 4 of this thesis each address separate questions relating to the scaling behavior of stochastic processes.

The problem of constructing reasonable stochastic processes of specified and varying roughness has recently received significant attention. As noted earlier the Hölder exponent of fractional Brownian motions is equal to H almost surely. This raised the question of whether a stochastic process could be found where the Hölder exponent varies in time. Multifractional Brownian motion was introduced as a Gaussian process with sample paths that have Hölder exponent $H(t)$ at time t almost surely for suitable functions $H(t)$. Other variants have been developed to expand the class of functions $H(t)$. The most general is generalized multifractional Brownian motion which can have any function as its Hölder exponent provided the function is a limit infimum of a sequence of continuous functions taking values in $[a, b]$ for some fixed $0 < a < b < 1$.

Understandably in constructing multifractional Brownian motion and its variants most attention has been focused on its local behavior. These constructions, however, have neglected the larger scale properties of the process. Even for a smooth function like $H(t) = \frac{1}{2} + \frac{1}{4}\cos(\frac{\pi t}{2})$ multifractional Brownian motion can have extreme oscillations when t is large.

Chapter 2 identifies the problem of large changes in magnitude and explains why they occur. We decompose multifractional Brownian motion into the sum of a smooth process and an integral of fractional white noise. The smooth process clearly does not affect the roughness but it is highly correlated and is responsible for the large changes in magnitude. So by instead taking just the integral of fractional white noise we retain the local properties without the unwanted fluctuations on the large scale. We call it integrated fractional white noise. We show that it is well defined and has the specified Hölder exponent. We also show how to estimate the Hölder exponent with our model. Our estimator is strongly consistent and under mild conditions asymptotically normal. Integrated fractional white noise has other advantages including that it can be extended to piecewise continuous $H(t)$ in a natural way. Also if $H(t)$ is taken to be a stationary random process then our process has stationary increments. All these advantages promise to make integrated fractional white noise significantly more useful from a modelling perspective.

Chapter 3 investigates the scaling properties of returns from financial data. The traditional Black and Scholes model of stock returns is Brownian motion which is of course self-similar. There is, however, growing empirical evidence that real returns exhibit more complex scaling behavior. This chapter compares two competing models: Brownian motion in multifractal time from [Mandelbrot et al., 1997] and [Mandelbrot, 2001a] and fractal activity time geometric Brownian motion from [Heyde, 1999].

Multifractal stochastic processes are a recent generalization of multifractal measures. A process is multifractal if its moments scale as $E|X(t)|^q = t^{\tau(q)}E|X(1)|^q$ where τ is called the scaling function. Self-similar processes are multifractal with linear scaling function. Other multifractal processes have concave scaling functions.

The main evidence for accepting a multifractal model rather than a simpler self-similar one is that the estimated scaling functions is concave. This method is immediately suspect as it involves estimating moments which may not exist. We show that this has an alternative explanation, that it is caused by the heavy tails of the returns and that after removing the largest observations the scaling function is close to linear. We prove a limit theorem which shows that some asymptotically self-similar heavy tailed processes can appear to have concave scaling functions. We conclude that it is not necessary to model returns as multifractals.

Fractal activity time geometric Brownian motion models the activity time as an asymptotically self-similar process. The evidence is based on the estimated scaling function of the squares of the returns which is consistent with long range dependence as self-similarity. The concentration of extreme values of the activity time suggested in [Heyde and Leonenko, 2005] is investigated and this leads onto the limit theorem proved in Chapter 4.

Chapter 4 extends Theorem 1.1.8 to functions f such that $f(X_1)$ is in the domain of attraction of a stable non-Gaussian law with finite mean. For general processes both long range dependence and heavy tails affect the rate of convergence and we find that for this sequence one of them dominates the other depending on the choice of parameters.

Because the sequence no longer has finite variance the usual methods using its Hermite decomposition can not be used directly. Instead we take a conditional expectation which does have finite variance and then apply Theorem 1.1.8. The remainder of the process is then shown to converge to α-stable Lévy motion. A critical element of this proof is to show that the extreme values of the sequence do not occur in clusters and instead are asymptotically independently. The result is then applied to the fractal activity time of [Heyde and Leonenko, 2005] to prove convergence in the case where the activity time has infinite variance.

Chapter 2

Integrated Fractional White Noise

While fractional Brownian motion has constant Hölder exponent various applications, for example terrain synthesizing, call for fractal properties with non-constant Hölder exponent. Multifractional Brownian motion was developed to produce sample paths with varying Hölder exponent. However, we will show that multifractional Brownian motion is very sensitive to changes in the selected Hölder exponent and has extreme changes in magnitude. We suggest an alternative stochastic process called Integrated Fractional White Noise which retains the important local properties but avoids the undesirable oscillations in magnitude. We also show how the Hölder exponent can be estimated locally from discrete data in this model.

2.1 Introduction

Fractional Brownian motion has been used to synthesize the roughness and fractal properties found in geographical formations (e.g. see [Voss, 1985]). While locally this may be a good approximation the irregularity of the surface may vary due to geological processes such as erosion. Multifractional Brownian motion (MBM) was introduced in order to model such processes where the local roughness varies. It was introduced in [Peltier and Lévy Véhel, 1995], based on the integral moving average representation of FBM, as the stochastic integral

$$M_H = \frac{1}{\Gamma(H(t)+\frac{1}{2})} \int_{-\infty}^{t} (t-u)^{H(t)-\frac{1}{2}}\, W(du) - \int_{-\infty}^{0} (-u)^{H(t)-\frac{1}{2}}\, W(du)$$

where W is a Wiener measure. The function $H(t)$ is continuous and $0 < H(t) < 1$. A second version was introduced in [Benassi et al., 1997] using the harmonizable integral representation of FBM

$$M_H(t) = \mathrm{Re} \int_{\mathbb{R}} \left(\frac{e^{i\xi t} - 1}{|\xi|^{H(t)+1/2}} \right) \widetilde{W}(d\xi)$$

where $\widetilde{W} = W_1 + iW_2$ and W_1 and W_2 are independent Wiener measures. When $H(t)$ is constant both versions are simply FBM. These two MBM versions do not have the same covariance structure. More generally MBM is defined as $M_H(t) := B_{H(t)}(t)$ where $B_H(t)$ is a family of fractional Brownian motions which is continuous in both t and H. While there is only one fractional Brownian motion for each H it is shown in [Stoev and Taqqu, 2004a] that there is a large class of families of FBMs with nontrivially different covariance structures.

Driving the definition of MBM is that its Hölder exponent can be specified at each point. The regularity assumptions that the Hölder exponent of $H(t)$ is greater than the value of $H(t)$ for all t is made. Under this assumption the Hölder exponent of $M_H(t)$ is $H(t)$ almost surely. The processes are also locally asymptotically self-similar. A process is locally asymptotically self-similar at t with parameter H if

$$\left(\frac{M_H(t+sh) - M_H(t)}{h^H}\right) \xrightarrow{d} V(s)$$

where $V(s)$ is the self-similar tangent process. The tangent process for MBM is FBM.

Variations on MBM have been proposed in order to expand the class of functions $H(t)$ for which it can be defined. Step multifractional Brownian motion was introduced in [Benassi et al., 1999] to extend MBM to the seemingly simple case of jump discontinuities using a wavelet decomposition. Generalized Multifractional Brownian motion takes a sequence of functions $(H_n(t))$ as parameters and has Hölder exponent $H(t) = \liminf H_n(t)$. It greatly expands the possible class of Hölder exponents (see [Ayache and Lévy Véhel, 2004]).

When MBM has non-constant $H(t)$ it follows from the local self-similarity property that it does not have stationary increments. This is to be expected as the aim of MBM was to have roughness, and therefore increments varying in time. As MBM is a Gaussian process its covariance structure completely determines its finite dimensional distributions. Its variance is given by $EM_H(t)^2 = t^{2H(t)}$ and so if $H(t)$ decreases over time the variance can actually decrease. Also the variance of the increments can vary greatly when t is large. By the triangle inequality under the L^2 norm the variance of the increment $M_H(t+1) - M_H(t)$ is greater than or equal to $|(t+1)^{H(t+1)} - t^{H(t)}|^2$ independent of the family of FBM chosen. When t is large this estimate can be very large depending on the change in $H(t)$. For example Figure 2.1 shows a typical sample path of MBM with $H(t) = \frac{1}{2} + \frac{1}{4}\cos(\frac{\pi t}{2})$ and Figure 2.2 shows a typical sample path of MBM with $H(t) = \frac{1}{2} + \frac{1}{10}\cos(5\pi t)$. An unfortunate consequence of MBM is that in specifying the local Hölder coefficient the variance is also specified to have dramatic changes in magnitude when t is large.

While we noted that varying local self-similarity and Hölder exponent is not compatible with stationary increments we propose a less restrictive alternative.

2.1.1 Definition ($\mathcal{H}$-Stationary Increments)**.** Let $\mathcal{H}$ be a class of continuous functions mapping $\mathbb{R}$ into $(0,1)$, closed under translations and let $\mathcal{S}$ be a space of stochastic processes. Then we say the function $\mathcal{M} : \mathcal{H} \mapsto \mathcal{S}$ has $\mathcal{H}$-stationary increments if when $H_1, H_2 \in \mathcal{H}$ and for some fixed u, $H_1(t) = H_2(t+u)$ for all $t \in [a, a+b]$ then

$$\{\mathcal{M}(H_1)(a+t) - \mathcal{M}(H_1)(a)\}_{t\in[0,b]} \stackrel{d}{=} \{\mathcal{M}(H_2)(a+u+t) - \mathcal{M}(H_2)(a+u)\}_{t\in[0,b]}$$

where equality is in finite dimensional distributions restricted to the interval $t \in [0, b]$.

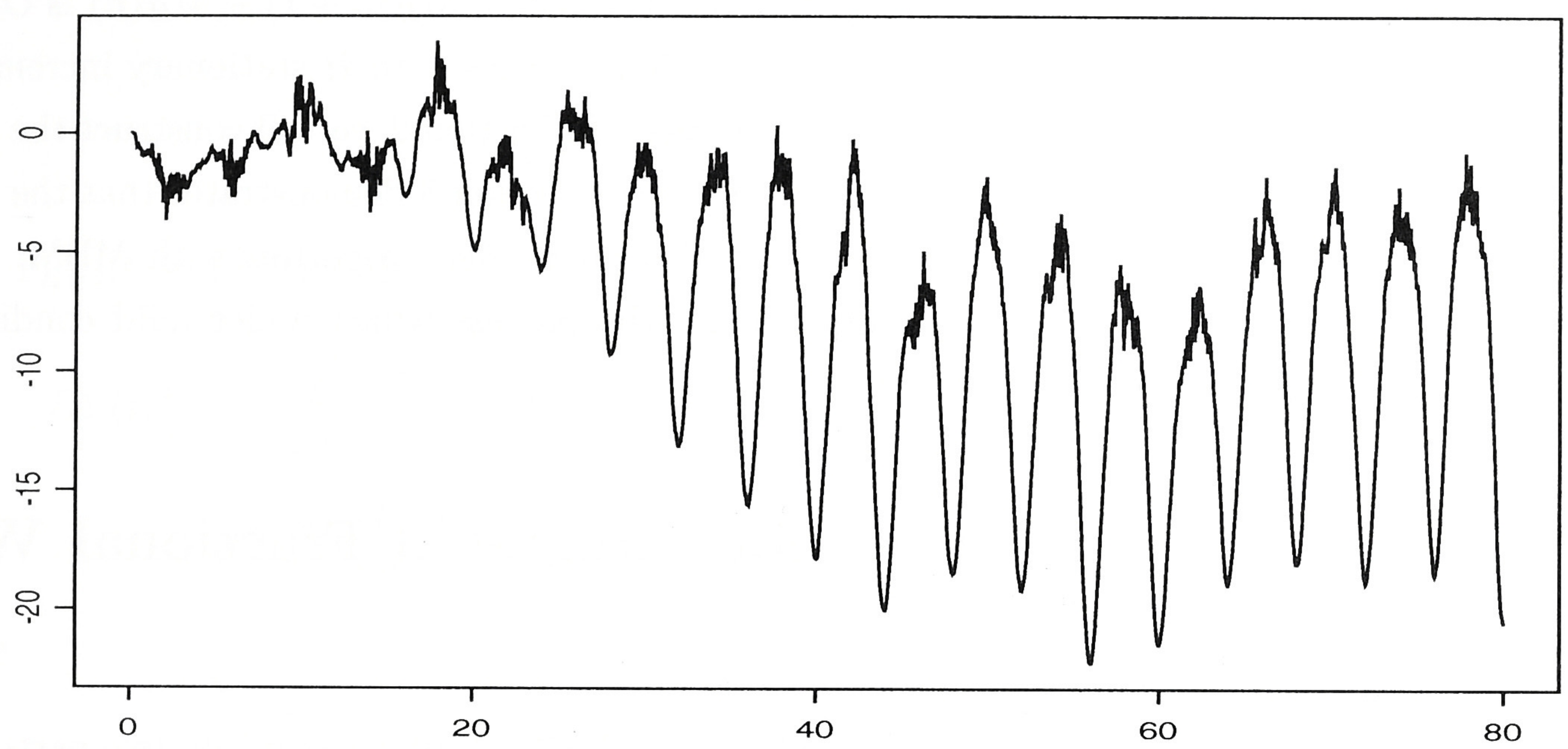

Figure 2.1: Multifractional Brownian motion with $H(t) = \frac{1}{2} + \frac{1}{4}\cos(\frac{\pi t}{2})$.

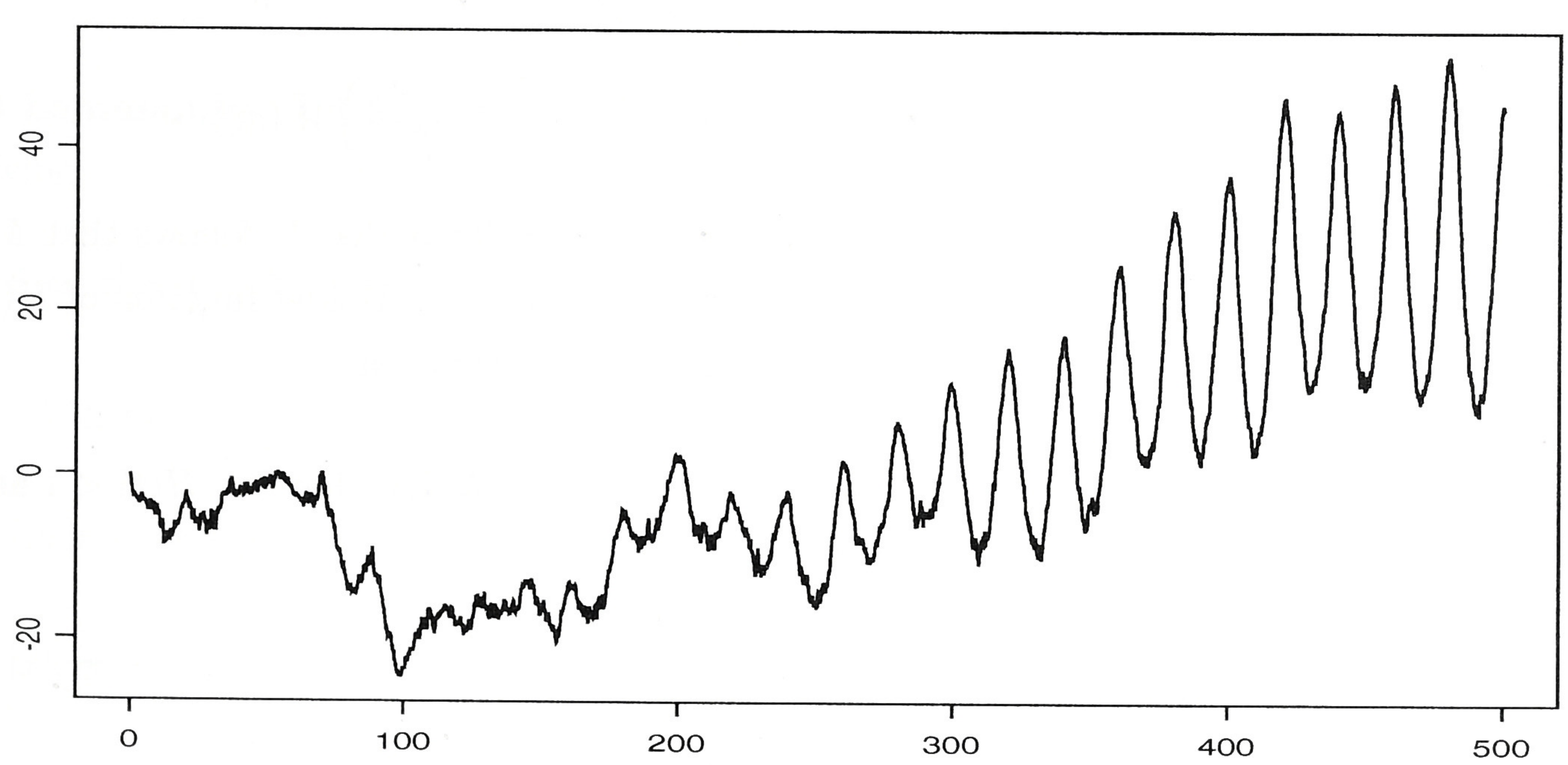

Figure 2.2: Multifractional Brownian motion with $H(t) = \frac{1}{2} + \frac{1}{10}\cos(5\pi t)$.

This definition means that the distribution of the increment $\mathcal{M}(H)(a+b) - \mathcal{M}(H)(a)$ depends only on the value taken by $H(t)$ in the interval $[a, a+b]$ and not on the relative position of the interval. Multifractional Brownian motion does not have $\mathcal{H}$-stationary increments. When $H(t) = \frac{1}{2} + \frac{1}{4}\cos(\pi t)$ the variance of the increments $M_H(n+1) - M_H(n)$ is $O(n^{\frac{3}{2}})$.

This chapter introduces a class of stochastic processes with $\mathcal{H}$-stationary increments as an alternative to multifractional Brownian motion. In Section 2 we will construct the process and establish a formula for its covariance structure. Section 3 demonstrates that the process shares the local asymptotic self-similarity and local Hölder properties with MBM. Section 4 examines a strongly consistent estimator on the process which under mild conditions is asymptotically normally distributed.

2.2 Variance Estimate for Integrated Fractional White Noise

In order to motivate the alternative definition of MBM we will break it into two parts. While MBM is obviously not differentiable it can be differentiated as a stochastic process in the space of stochastic distributions. Assuming that $H(t)$ is continuously differentiable,

$$\frac{d}{dt} M_H(t) = H'(t) \frac{\partial}{\partial H} B_H(t) + W_H(t), \tag{2.1}$$

where $W_H(t)$ is fractional white noise as in [Elliott and van der Hoek, 2003]. Unlike fractional white noise the term $\frac{\partial}{\partial H} B_H(t)$ is a Gaussian random variable with variance bounded on compacts. It becomes very large when t is large. Then

$$\begin{aligned} M_H^{(f)}(t) &:= \int_0^t H'(s) \frac{\partial}{\partial H} B_H(s) ds \\ &= \int_0^t H'(s) \mathrm{Re} \int_{\mathbb{R}} \left(\frac{\ln(|\xi|)(e^{i\xi t} - 1)}{|\xi|^{H(t)+1/2}} \right) \widetilde{W}(d\xi) ds \end{aligned}$$

is a Gaussian finite variation process with locally Lipschitz paths. It follows that $M_H(t) - M_H^{(f)}(t)$ is also locally asymptotically self-similar and has local Hölder function of $H(t)$. It is also has $\mathcal{H}$-stationary increments and so motivates our definition.

2.2.1 Definition (Integrated Fractional White Noise (IFWN)). For $0 < H(t) < 1$ and $H(t)$ continuous define integrated fractional white noise as

$$Y(t) = \int_0^t W_{H(s)}(s) ds \tag{2.2}$$

or equivalently

$$Y_{H(t)}(t) = \mathrm{Re} \int_{\mathbb{R}} \left(\int_0^t \frac{i\xi e^{i\xi s}}{|\xi|^{H(s)+1/2}} ds \right) \widetilde{W}(d\xi). \tag{2.3}$$

This process is not a priori in L^2 and so this needs to be estimated. In fact when $H(t)$ is less than $\frac{1}{2}$ greater regularity must be assumed.

2.2.2 Theorem. Suppose $H(t)$ is continuous and $0 < a \leq H(t) \leq b < 1$ and that there exists some $\beta, C_1 > 0$ such that

$$\beta + a > \frac{1}{2} \tag{2.4}$$

and

$$|H(t) - H(s)| \leq C_1|t-s|^\beta. \tag{2.5}$$

Then $Y_{H(t)}(t) \in L^2(\Omega)$ with

$$\begin{aligned} EY(t)^2 &= \int_0^t A(H(s))H(s)s^{2H(s)-1}ds + \int_0^t A(H(s))H(s)(t-s)^{2H(s)-1}ds \\ &+\frac{1}{4}\int_0^t\int_0^t f_H(x,y)dxdy < \infty \end{aligned} \tag{2.6}$$

where

$$A(h) = \int_{\mathbb{R}} \left| \int_0^1 \frac{i\xi e^{i\xi s}}{|\xi|^{h+1/2}} ds \right|^2 d\xi$$

and

$$\begin{aligned} f_H(x,y) &= 2A(\frac{H(x,y)}{2})(H(x,y))(H(x,y)-1)|x-y|^{H(x,y)-2} \\ &-A(H(x))(2H(x))(2H(x)-1)|x-y|^{2H(x)-2} \\ &-A(H(y))(2H(y))(2H(y)-1)|x-y|^{2H(y)-2} \end{aligned}$$

with $H(x,y) = H(x) + H(y)$.

We first establish an estimate for $f_H(x,y)$.

2.2.3 Lemma. Let $f \in C^2[a,b]$ and let $c_1, c_2 > 0$ be constants. Then for any N, r, s, x, y, z satisfying

1. $0 < |z| < 1 < N$.
2. $|z|^s \leq c_1$.
3. $(\ln|z|)(|x| + |y|) \leq c_2$.
4. $|x|, |y| \leq C_1 N^{-\beta}$.
5. $r + ix + jy \in [a,b]$ for $i = 0,1$ and $j = 0,1$.

Then there exists a constant $K > 0$ depending only on f, c_1, c_2, β, C_1 such that

$$\left| \sum_{i=0}^1 \sum_{j=0}^1 (-1)^{i+j} f(r + ix + jy)|z|^{s+ix+jy} \right| \leq K(1 + \ln|z|)^2 N^{-2\beta}.$$

Proof. (Lemma 2.2.3) By expanding in Taylor series

$$\begin{aligned}
&\sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}f(r+ix+jy)|z|^{s+ix+jy}\\
=\ &\sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}(f(r)+f'(r)(ix+jy)+\frac{1}{2}f''(r_{ij})(ix+jy)^2)\\
&\cdot|z|^s(1+(\ln|z|)(ix+jy)+\frac{1}{2}e^{s_{ij}}(\ln|z|)^2(ix+jy)^2)\\
=\ &|z|^s\sum_{i=0}^{1}\sum_{j=0}^{1}(-1)^{i+j}\frac{1}{2}e^{s_{ij}}f(r)(\ln|z|)^2(ix+jy)^2\\
&+f'(r)(ix+jy)((\ln|z|)(ix+jy)+\frac{1}{2}e^{s_{ij}}(\ln|z|)^2(ix+jy)^2)\\
&+\frac{1}{2}f''(r_{ij})(ix+jy)^2(1+(\ln|z|)(ix+jy)+\frac{1}{2}e^{s_{ij}}(\ln|z|)^2(ix+jy)^2)
\end{aligned}$$

where $r_{ij}\in[a,b]$ and $|s_{ij}|\le c_2$ and the result follows. □

2.2.4 Lemma. For any $-1<\alpha<2a+2\beta-2$ there exists C_2 depending only on a,b,β,C_1,ϵ such that

$$|f_H(x,y)|\le C_2\min\{|x-y|,1\}^{\alpha}. \tag{2.7}$$

In particular this implies that $f_H(x,y)\in L^1([0,t]^2)$.

Proof. (Lemma 2.2.4) First note that $A(h)$ is C^∞ on $[a,b]$. When $|x-y|>1$ the bound clearly exists. When $|x-y|\le 1$ the result follows from Lemma 2.2.3. □

Proof. (Theorem 2.2.2) Begin by approximating H by step functions so that

$$H_n(s)=\sum_i \chi_{[i/2^n,(i+1)/2^n)}H(i/2^n)$$

and

$$Y_n(t)=\int_{\mathbb{R}}\left(\int_0^t\frac{i\xi e^{i\xi s}}{|\xi|^{H_n(s)+1/2}}ds\right)\widetilde{W}(d\xi). \tag{2.8}$$

Since Y_n is the sum of increments of fractional Brownian motion it is in $L^2(\Omega)$. Now with $H_1=H_n(i/2^n)$ and $H_2=H_n(j/2^n)$

$$\begin{aligned}
&E(Y_n((j+1)/2^n)-Y_n(j/2^n))(Y_n((k+1)/2^n)-Y_n(k/2^n))\\
=\ &\mathrm{Re}\int_{\mathbb{R}}\left(\int_{j/2^n}^{(j+1)/2^n}\frac{i\xi e^{i\xi s}}{|\xi|^{H_1+1/2}}ds\right)\overline{\left(\int_{k/2^n}^{(k+1)/2^n}\frac{i\xi e^{i\xi s}}{|\xi|^{H_2+1/2}}ds\right)}d\xi\\
=\ &\mathrm{Re}\int_{\mathbb{R}}\frac{e^{i(j-k)2^{-n}\xi}|e^{2^{-n}i\xi}-1|^2}{|\xi|^{H_1+H_2-1}}d\xi\\
=\ &A(\frac{H_1+H_2}{2})\frac{1}{2}2^{-n(H_1+H_2)}[|i-j+1|^{H_1+H_2}-2|j-k|^{H_1+H_2}+|j-k-1|^{H_1+H_2}]\\
\approx\ &A(\frac{H_1+H_2}{2})\frac{1}{2}2^{-n(H_1+H_2)}(H_1+H_2)(H_1+H_2-1)|j-k|^{H_1+H_2-2}
\end{aligned}$$

and so we can estimate

$$\begin{aligned}
EY_n(t)^2 &= \sum_{i=1}^{\lfloor 2^n t\rfloor}\sum_{j=1}^{\lfloor 2^n t\rfloor} A\Big(\frac{H(\frac{i}{2^n},\frac{j}{2^n})}{2}\Big)\frac{1}{2}2^{-n(H(\frac{i}{2^n},\frac{j}{2^n}))}[|i-j+1|^{H(\frac{i}{2^n},\frac{j}{2^n})} \\
&\quad -2|i-j|^{H(\frac{i}{2^n})+H(\frac{j}{2^n})} + |i-j-1|^{H(\frac{i}{2^n})+H(\frac{j}{2^n})}] \\
&= \sum_{i=1}^{\lfloor 2^n t\rfloor}\sum_{j=1}^{\lfloor 2^n t\rfloor} A(H(\frac{i}{2^n}))\frac{1}{2}2^{-n2H(\frac{i}{2^n})}[|i-j+1|^{2H(\frac{i}{2^n})} \\
&\quad -2|i-j|^{2H(\frac{i}{2^n})} + |i-j-1|^{2H(\frac{i}{2^n})}] \\
&\quad +A\Big(\frac{H(\frac{i}{2^n},\frac{j}{2^n})}{2}\Big)\frac{1}{2}2^{-n(H(\frac{i}{2^n},\frac{j}{2^n}))}[|i-j+1|^{H(\frac{i}{2^n},\frac{j}{2^n})} \\
&\quad -2|i-j|^{H(\frac{i}{2^n},\frac{j}{2^n})} + |i-j-1|^{H(\frac{i}{2^n},\frac{j}{2^n})}] \\
&\quad -A(H(\frac{i}{2^n}))\frac{1}{4}2^{-n2H(\frac{i}{2^n})}[|i-j+1|^{2H(\frac{i}{2^n})} - 2|i-j|^{2H(\frac{i}{2^n})} + |i-j-1|^{2H(\frac{i}{2^n})}] \\
&\quad -A(H(\frac{j}{2^n}))\frac{1}{4}2^{-n2H(\frac{j}{2^n})}[|i-j+1|^{2H(\frac{j}{2^n})} - 2|i-j|^{2H(\frac{j}{2^n})} + |i-j-1|^{2H(\frac{j}{2^n})}] \\
&= (I) + (II).
\end{aligned}$$

By the Dominated Convergence Theorem

$$\begin{aligned}
(I) &= \sum_{i=1}^{\lfloor 2^n t\rfloor} \frac{1}{2}2^{-n2H(\frac{i}{2^n})}[|i|^{2H(\frac{i}{2^n})} - 2|i-1|^{2H(\frac{i}{2^n})} \\
&\quad -2|i-2^n|^{2H(\frac{i}{2^n})} + |i-2^n-1|^{2H(\frac{i}{2^n})}] \\
&\to \int_0^t A(H(s))H(s)s^{2H(s)-1}ds + \int_0^t A(H(s))H(s)(1-s)^{2H(s)-1}ds
\end{aligned}$$

as $n \to \infty$. Another application of the Dominated Convergence Theorem shows that

$$(II) \to \frac{1}{4}\int_0^t\int_0^t f_H(x,y)dxdy.$$

Similar calculations in estimating $E(Y_n - Y_m)^2$ show that Y_n is a Cauchy sequence in $L^2(\Omega)$. Finally

$$\int_0^t \frac{i\xi e^{i\xi s}}{|\xi|^{H_n(s)+1/2}}ds \to \int_0^t \frac{i\xi e^{i\xi s}}{|\xi|^{H(s)+1/2}}ds$$

pointwise implies that $Y_n \to Y$ in $L^2(\Omega)$. $\square$

Corollary 2.2.5 follows immediately from Theorem 2.2.2.

2.2.5 Corollary. The value of $E(Y(t_2) - Y(t_1))^2$ depends only on the values of $H(t)$ for $t \in [t_1, t_2]$ and is given by

$$\begin{aligned}
E(Y(t_2) - Y(t_1))^2 &= \int_{t_1}^{t_2} A(H(s))H(s)(s-t_1)^{2H(s)-1}ds \\
&\quad + \int_{t_1}^{t_2} A(H(s))H(s)(t_2-s)^{2H(s)-1}ds \\
&\quad + \frac{1}{4}\int_{t_1}^{t_2}\int_{t_1}^{t_2} f_H(x,y)dxdy.
\end{aligned}$$

Further

$$\begin{aligned}
EY(t_1)Y(t_2) &= \frac{1}{2}\Big[\int_0^{t_1} A(H(s))H(s)s^{2H(s)-1}ds + \int_0^{t_1} A(H(s))H(s)(t_1-s)^{2H(s)-1}ds \\
&\quad + \int_0^{t_2} A(H(s))H(s)s^{2H(s)-1}ds + \int_0^{t_2} A(H(s))H(s)(t_2-s)^{2H(s)-1}ds \\
&\quad - \int_{t_1}^{t_2} A(H(s))H(s)(s-t_1)^{2H(s)-1}ds - \int_{t_1}^{t_2} A(H(s))H(s)(t_2-s)^{2H(s)-1}ds\Big] \\
&\quad + \frac{1}{4}\int_0^{t_1}\int_0^{t_2} f_H(x,y)dxdy. \qquad (2.9)
\end{aligned}$$

The next two corollaries are established by modifying the proof of Theorem 2.2.2 under conditions that imply that

$$\begin{aligned}
&E(Y(t_2)-Y(t_1))(Y(t_4)-Y(t_3)) \\
&= \sum_{i=1+\lfloor 2^n t_2\rfloor}^{\lfloor 2^n t_2\rfloor} \sum_{j=1+\lfloor 2^n t_3\rfloor}^{\lfloor 2^n t_4\rfloor} A(\frac{H(\frac{i}{2^n},\frac{j}{2^n})}{2})\frac{1}{2}2^{-n(H(\frac{i}{2^n},\frac{j}{2^n}))}[|i-j+1|^{H(\frac{i}{2^n},\frac{j}{2^n})} \\
&\quad -2|i-j|^{H(\frac{i}{2^n})+H(\frac{j}{2^n})} + |i-j-1|^{H(\frac{i}{2^n})+H(\frac{j}{2^n})}] \\
&\to \frac{1}{2}\int_{t_1}^{t_2}\int_{t_3}^{t_4} A(\frac{H(x,y)}{2})(H(x,y))(H(x,y)-1)|x-y|^{H(x,y)-2}dxdy.
\end{aligned}$$

2.2.6 Corollary. When $a > \frac{1}{2}$ the estimate of $EY(t)^2$ is given by

$$\frac{1}{2}\int_0^t\int_0^t A(\frac{H(x,y)}{2})(H(x,y))(H(x,y)-1)|x-y|^{H(x,y)-2}dxdy.$$

When $H(x) < \frac{1}{2}$ this function is not integrable.

2.2.7 Corollary. When $t_1 < t_2 \le t_3 < t_4$ the estimate of $E(Y(t_2)-Y(t_1))(Y(t_4)-Y(t_3))$ is given by

$$\frac{1}{2}\int_{t_1}^{t_2}\int_{t_3}^{t_4} A(\frac{H(x,y)}{2})(H(x,y))(H(x,y)-1)|x-y|^{H(x,y)-2}dxdy$$

irrespective of a.

2.2.8 Proposition. [$\mathcal{H}$-Stationary Increments] Let $\mathcal{H}$ be the class of continuous functions such that $|H(t)-H(s)| \le C_1|t-s|^\beta$ and let $\mathcal{M} : \mathcal{H} \mapsto \mathcal{S}$ map $H \in \mathcal{H}$ to $Y_H(t)$. Then $\mathcal{M}$ has $\mathcal{H}$-stationary increments.

Proof. Let $H_1, H_2 \in \mathcal{H}$ be such that for some u and interval $[a, a+b]$, $H_1(t) = H_2(t+u)$ for all $t \in [a, a+b]$. Then

$$\begin{aligned}
&\{\mathcal{M}(H_1)(a+t) - \mathcal{M}(H_1)(a)\}_{t\in[0,b]} \\
&\stackrel{d}{=} \{\mathrm{Re}\int_{\mathbb{R}}\left(\int_0^t \frac{i\xi e^{i\xi(a+s)}}{|\xi|^{H(s)+1/2}}ds\right)\widetilde{W}(d\xi)\}_{t\in[0,b]} \\
&\stackrel{d}{=} \{\mathrm{Re}\int_{\mathbb{R}}\left(\int_0^t \frac{i\xi e^{i\xi(a+u+s)}}{|\xi|^{H(s)+1/2}}ds\right)\widetilde{W}(d\xi)\}_{t\in[0,b]} \\
&\stackrel{d}{=} \{\mathcal{M}(H_2)(a+u+t) - \mathcal{M}(H_2)(a+u)\}_{t\in[0,b]}
\end{aligned}$$

and so $\mathcal{M}$ has $\mathcal{H}$-stationary increments. $\square$

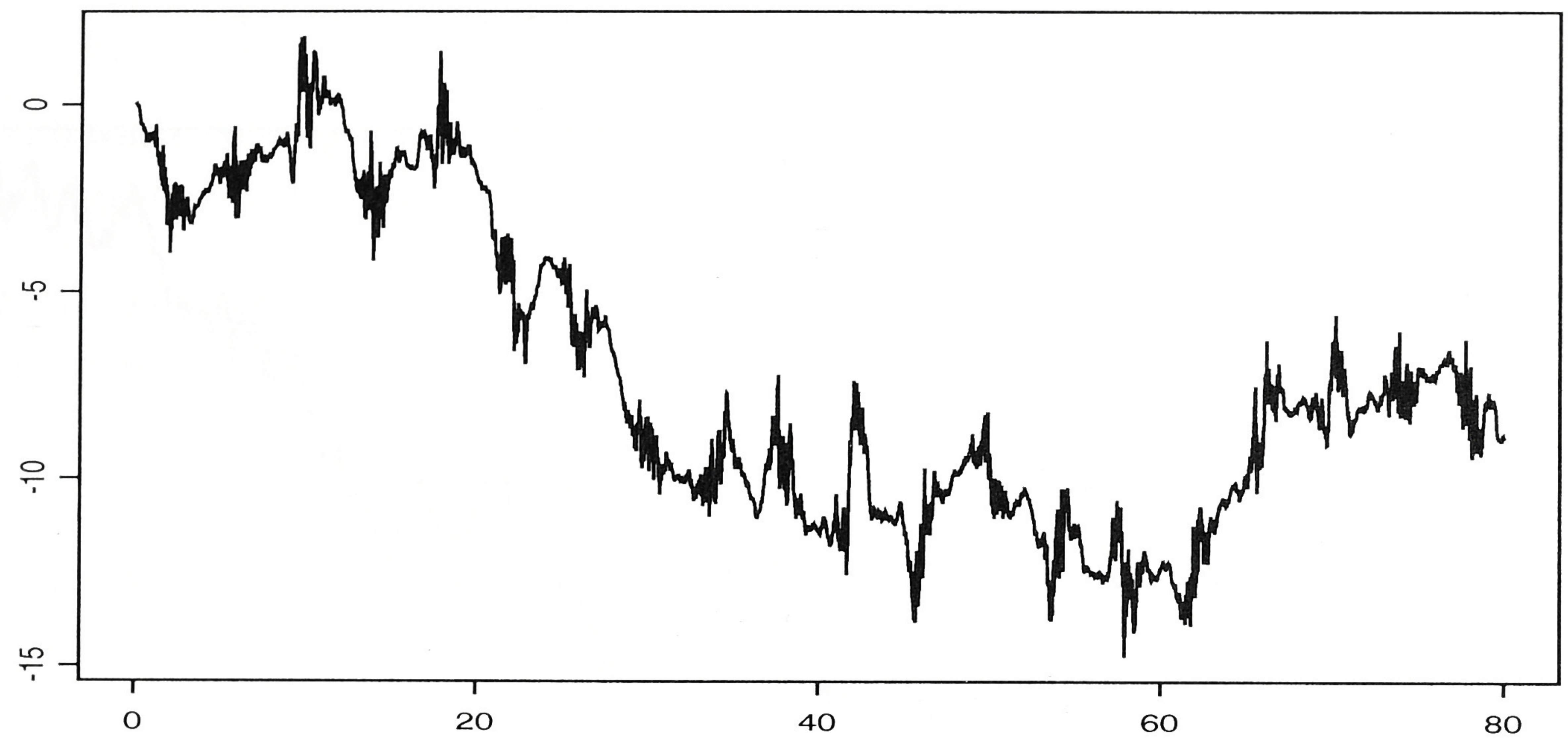

Figure 2.3: $Y_H(t)$ with $H(t) = \frac{1}{2} + \frac{1}{10}\cos(5\pi t)$.

This result shows that the increments depend only on the local values of $H(t)$ and not on the position of the increment in time. It is also important for the generalization when $H(t)$ is random. Suppose $H(t)$ is a stationary process whose paths are in $\mathcal{H}$. Then it follows from $\mathcal{H}$-stationary increments that $\mathcal{M}(H)(t)$ has stationary increments.

Figure 2.3 shows a typical sample path of $Y_H(t)$ with $H(t) = \frac{1}{2} + \frac{1}{4}\cos(\frac{\pi t}{2})$ while Figure 2.4 shows a typical sample path of $Y_H(t)$ with $H(t) = \frac{1}{2} + \frac{1}{10}\cos(5\pi t)$. Unlike Figures 2.1 and 2.2 there are no extreme swings in magnitude when t is large.

The definition of $Y_H(t)$ also naturally extends to piecewise continuous functions. Unlike MBM this does not lead to discontinuities. Figure 2.5 shows a typical sample path of $Y_H(t)$ with

$$H(t) = \begin{cases} \frac{1}{4} & t \leq 10, \\ \frac{3}{4} & t > 10. \end{cases}$$

Throughout the rest of this chapter we will assume that $H(t)$ satisfies conditions (2.4) and (2.5).

2.3 Local Self-similarity and Path Properties

In this section we prove that IFWN retains the essential properties of MBM. That is it is locally self-similar at each time with Hurst parameter $H(t)$ and has Hölder Exponent $H(t)$ almost surely.

2.3.1 Theorem (Local Self-similarity). The process $Y_H(t)$ is locally self-similar about the

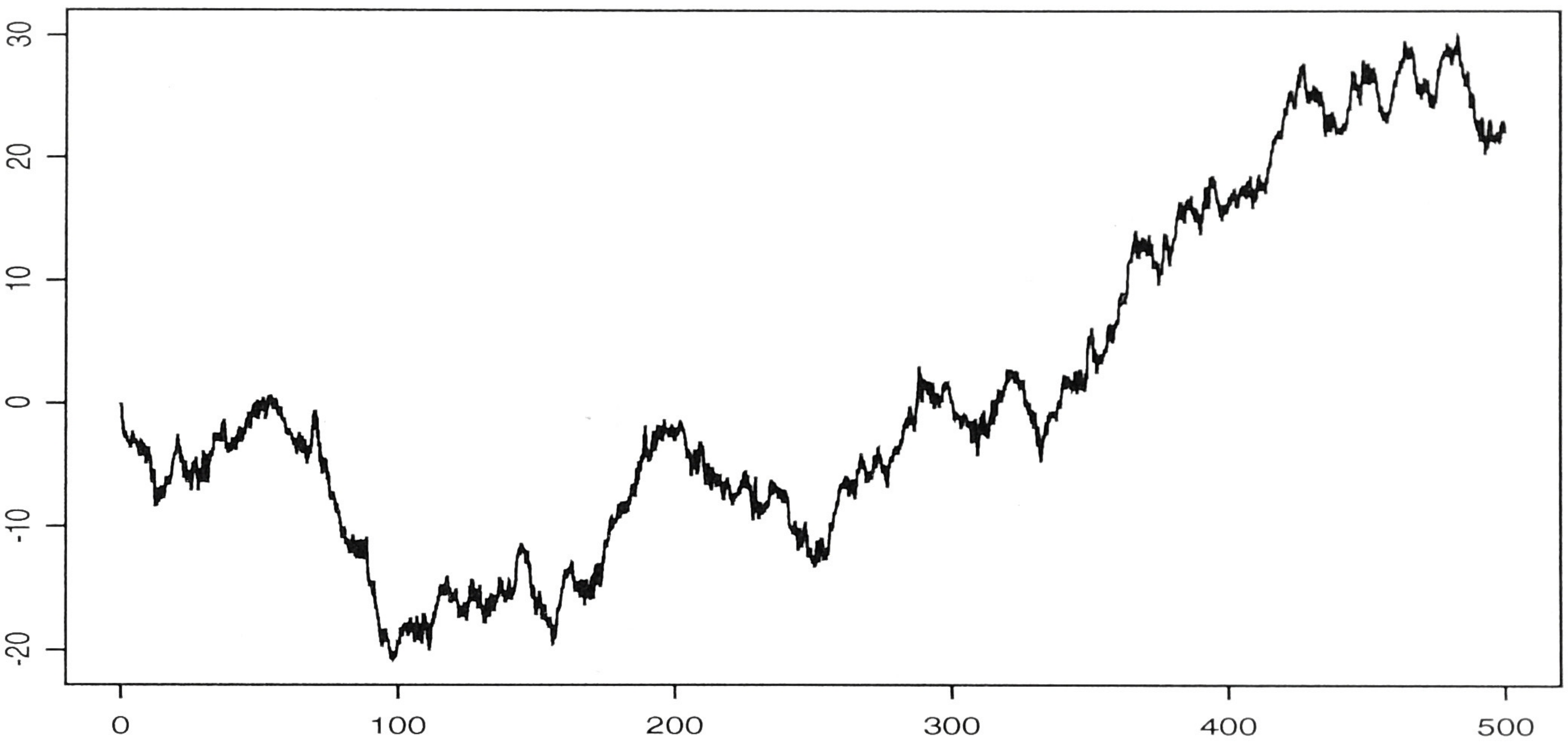

Figure 2.4: $Y_H(t)$ with $H(t) = \frac{1}{2} + \frac{1}{10}\cos(5\pi t)$.

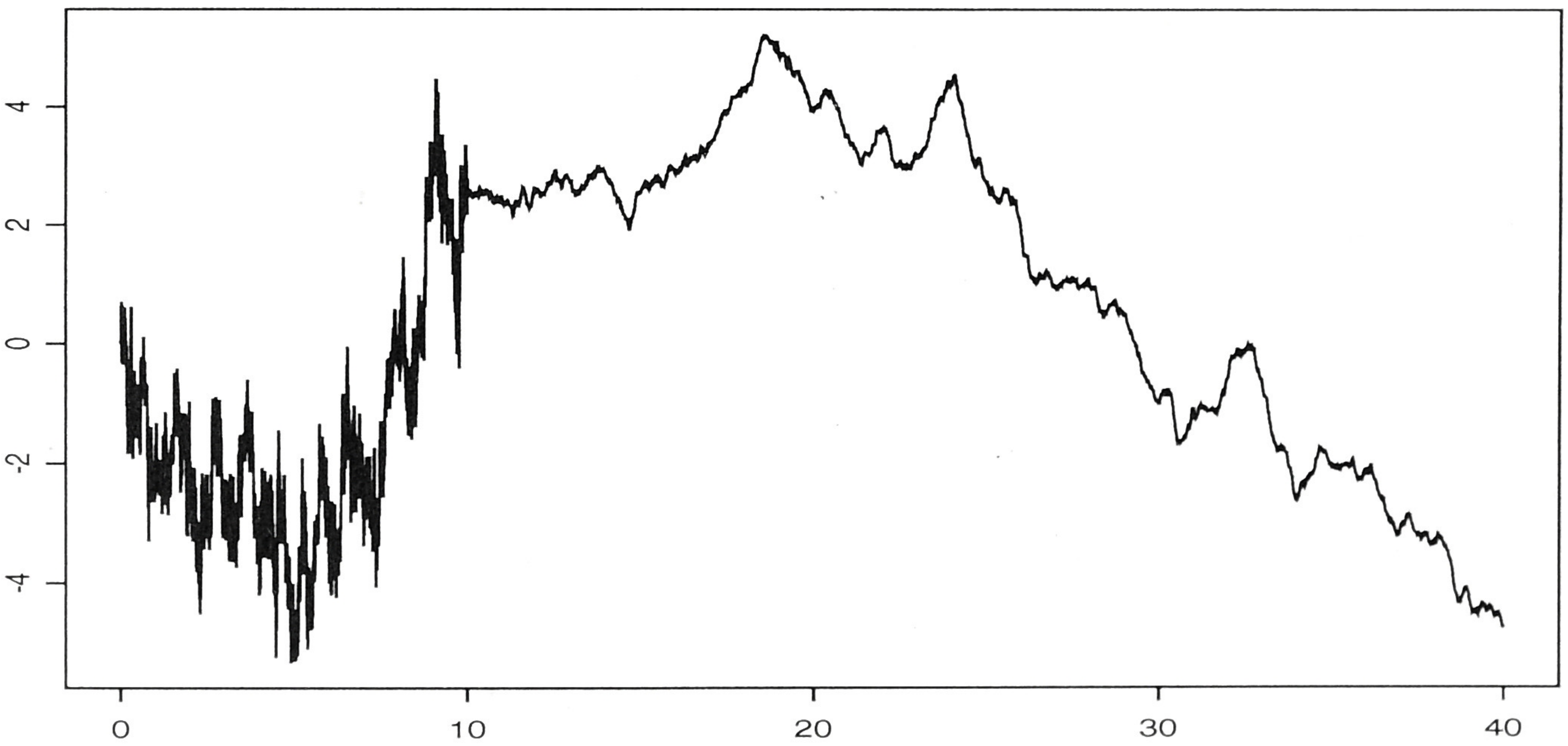

Figure 2.5: $Y_H(t)$ with discontinuous $H(t)$.

point t_0 with parameter $H(t_0)$. That is

$$\{h^{-H(t_0)}(Y_H(t_0+th)-Y_H(t_0))\} \stackrel{d}{=} \{A(H(t_0))B_{H(t_0)}(t)\}$$

as $h \to 0$ where convergence is in finite dimensional distributions and $B_{H(t_0)}(t)$ is fractional Brownian motion with parameter $H(t_0)$.

Proof. Without loss of generality let $t_0 = 0$. Since these are mean zero Gaussian processes it is sufficient to show that for each s, t

$$Eh^{-2H(0)}Y_H(sh)Y_H(th) \to A(H(0))EB_{H(0)}(s)B_{H(0)}(t).$$

We will show convergence of each integral in equation (2.9). By the Dominated Convergence Theorem,

$$\begin{aligned} h^{-2H(0)}\int_0^{th} A(H(x))H(x)s^{2H(x)-1}dx &= h^{-2H(0)}\int_0^{t} A(H(yh))H(yh)(yh)^{2H(yh)-1}hdy \\ &= \int_0^{t} A(H(yh))H(yh)y^{2H(yh)-1}h^{2(H(yh)-H(0))}dy \\ &\to \int_0^{t} \frac{1}{2}A(H(0))H(0)y^{2H(0)-1}du \\ &= A(H(0))t^{2H(0)} \end{aligned}$$

since $h^{2(H(yh)-H(0))} \to 1$ uniformly by the Hölder continuity of $H(t)$. The next five integrals follow similarly. By choosing a small neighborhood of 0 we can assume that $a < H(0) - \beta$ so that $2H(0) < \alpha + 2$ and so by Lemma 2.2.2

$$\begin{aligned} |h^{-2H(0)}\int_0^{t_1h}\int_0^{t_2h} f_H(x,y)dxdy| &\le h^{-2H(0)}C_2\int_0^{t_1h}\int_0^{t_2h} |x-y|^\alpha dxdy \\ &= h^{\alpha+2-2H(0)}\frac{C_2(t^{\alpha+2}+t^{\alpha+2}-(t-s)^{\alpha+2})}{(\alpha+1)(\alpha+2)} \to 0. \end{aligned}$$

Hence it follows that

$$Eh^{-2H(0)}Y_H(sh)Y_H(th) \to A(H(0))\frac{1}{2}[t^{2H(0)}+s^{2H(0)}-|t-s|^{2H(0)}] = A(H(0))EB_{H(0)}(s)B_{H(0)}(t)$$

as required. □

2.3.2 Proposition (Continuity). The process $Y_H(t)$ has a continuous version.

Proof. By Theorem 2.2.2 there exists a constant c such that

$$E(Y_H(t_1)-Y_H(t_2))^2 \le c|t_1-t_2|^{2a}$$

when $|t_1 - t_2|$ is small. Then

$$E(Y_H(t_1)-Y_H(t_2))^{\frac{2}{a}} \le c'|t_1-t_2|^2$$

and so by Kolmogorov's Continuity Theorem $Y_H(t)$ has a continuous version. □

2.3.3 Theorem (Hölder Exponent). With probability 1, $Y_H(t)$ has Hölder exponent $H(x)$ at x.

Proof. For any $\delta > 0$ we can choose $0 < \epsilon < 1$ so that $H(x) - \delta < a := \min\{H(t) : t \in [x-\epsilon, x+\epsilon]\}$. Then by Theorem 2.2.2 for $s, t \in [x-\epsilon, x+\epsilon]$

$$E|Y_H(t) - Y_H(s)|^2 \leq c_1|t-s|^{2a}$$

Choose k so that $a - (2k)^{-1} = H(x) - \delta$. Since Y_H is Gaussian

$$E|Y_H(t) - Y_H(s)|^{2k} \leq c|t-s|^{2ak}.$$

so by Kolmogorov's Continuity Theorem almost all sample paths of $Y_H(t)$ have Hölder continuity $H(x) - \delta$. Hence the Hölder exponent is at least $H(x)$. Fix $\gamma > H(x)$. Then by Theorem 2.3.1 we have that

$$E[|h|^{-\gamma}(Y_H(x+h) - Y_H(x))]^2 = O(h^{H(x)-\gamma})$$

and so

$$\limsup_{h\to 0} h^{-\gamma}|Y_H(x+h) - Y_H(x)| = \infty$$

almost surely so the Hölder exponent is $H(x)$. □

2.4 Identification of $H(t)$

For processes like fractional Brownian motion where the Hölder exponent is constant the Hölder exponent can be estimated by examining either local or long range properties. With every type of multifractional Brownian motion the Hölder exponent is a truly local property and must be estimated as such. The estimator used most frequently in the multifractional Brownian motion literature (for example see [Benassi et al., 2000],[Ayache and Lévy Véhel, 2004] and [Ayache et al., 2005]) is

$$\widehat{H}_N(t) = \frac{1}{2}\left(1 - \gamma - \frac{\ln V_N(t)}{\ln N}\right)$$

where $0 < \gamma < 1$ and

$$V_N(t) = \sum_{j=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor}\left(Y_H(t + \frac{j+1}{N}) - 2Y_H(t + \frac{j}{N}) + Y_H(t + \frac{j-1}{N})\right)^2.$$

While this estimator is strongly consistent it does not converge very quickly. Heuristically $EV_N(t) \approx C(t)N^{1-\gamma-2H}$ so

$$\frac{\ln V_N(t)}{\ln N} \approx \frac{\ln C(t)}{\ln N} + 1 - \gamma - 2H.$$

In fact we can show that

$$\widehat{H}_N(t) - H(t) = \epsilon_n - \frac{\ln C(t)}{2\ln N}.$$

By a similar method of proof to Theorem 2.4.9 we can show that if $(1-\gamma)(4H(t)-3) - 4\beta < 0$ then $cN^{(1-\gamma)/2}\epsilon_n$ converges in distribution to $N(0,1)$. However, the term $\frac{\ln C(t)}{\ln N}$ depends on

$H(t)$ and decays very slowly to 0 making $\widehat{H}_N(t)$ a very inefficient estimator of $H(t)$. We can prove better rate of convergence results for the estimator

$$\breve{H}_N(t) = \frac{1}{2}\left(\log_2 \frac{1+2\lfloor N^{1-\gamma}\rfloor}{1+2\lfloor (N/2)^{1-\gamma}\rfloor} + \log_2 \frac{V_{N/2}(t)}{V_N(t)}\right).$$

This estimator was used in [Benassi et al., 1998], albeit for a different process. We will prove consistency and a central limit theorem for this estimator but first some lemmas are required.

2.4.1 Lemma. Let $X_j = Y_H(t+\frac{j}{N}) - Y_H(t+\frac{j-1}{N})$. Then there exist constants $C_3, C_4, C_5, C_6 > 0$ such that for all $N > 1$ and $-\lfloor N^{1-\gamma}\rfloor \le i,j \le \lfloor N^{1-\gamma}\rfloor$

$$|E(X_{i+1}-X_i)^2 - (4-4^{H(t)})A(H(t))N^{-2H(t)}| \le C_3(\ln N)N^{-2H(t)-\gamma\beta}. \tag{2.10}$$

Further,

$$\begin{aligned}&\Big|E(X_{i+1}-X_i)(X_{j+1}-X_j) - \frac{1}{2}\Big(-|i-j+2|^{2H(t)} + 4|i-j+1|^{2H(t)}\\ &-6|i-j|^{2H(t)} + 4|i-j-1|^{2H(t)} - |i-j-2|^{2H(t)}\Big)A(H(t))N^{-2H(t)}\Big|\\ \le\ & C_4|i-j|^{2H(t)-2}(\ln(N))^2N^{-2H(t)-2\beta}\\ &+C_5|i-j|^{2H(t)-3}(\ln(N))^2N^{-2H(t)-\gamma\beta}.\end{aligned} \tag{2.11}$$

Also

$$\begin{aligned}&\frac{1}{2}A(H(t))\Big| - |i-j+2|^{2H(t)} + 4|i-j+1|^{2H(t)} - 6|i-j|^{2H(t)}\\ &+4|i-j-1|^{2H(t)} - |i-j-2|^{2H(t)}\Big| \le C_6|i-j|^{2H(t)-4}\end{aligned}$$

Proof. Let $a_N = \min\{H(t) : s\in[t-N^{-\gamma}, t+N^{-\gamma}]\}$, $b_N = \max\{H(t) : s\in[t-N^{-\gamma}, t+N^{-\gamma}]\}$, $A_{\min} = \min\{A(H(t)) : s\in[t-N^{-\gamma}, t+N^{-\gamma}]\}$ and $A_{\max} = \max\{A(H(t)) : s\in[t-N^{-\gamma}, t+N^{-\gamma}]\}$. By the Hölder condition on $H(t)$, $|a_N - H(t)| \le C_1N^{-\gamma\beta}$ and $|A_{\min} - A(H(t))| \le cN^{-\beta}$. Taking N large enough such that $2H(t) \le 2b_N < 2H(t)+\gamma\beta < \alpha+2$ then by Theorem 2.2.2,

$$\begin{aligned}E(X_{i+1}-X_i)^2 &= E(2X_{i+1}^2 + 2X_i^2 - (Y_H(t+\frac{j+1}{N}) - Y_H(t+\frac{j-1}{N}))^2)\\ &\le 4A_{\max}N^{-2a_N} - A_{\min}(N/2)^{-2b_N} + C_2N^{-\alpha-2}.\end{aligned}$$

Since $|A_{\max} - A(H(t))| \le cN^{-\gamma\beta}$ and $|N^{2H(t)-2a_N} - 1| \le c\ln(N)N^{-\gamma\beta}$,

$$E(X_{i+1}-X_i)^2 - (4-4^{H(t)})A(H(t))N^{-2H(t)} \le C_3(\ln N)N^{-2H(t)-\gamma\beta}.$$

The reverse inequality similarly holds proving equation (2.10). Now when $|i-j| > 2$ let

$$\begin{aligned}(I) &:= \int_{t+\frac{i}{N}}^{t+\frac{i+1}{N}}\int_{t+\frac{j}{N}}^{t+\frac{j+1}{N}}\Big|\sum_{k=0}^{1}\sum_{\ell=0}^{1}(-1)^{k+\ell}A(\frac{H(x-\frac{k}{N}, y-\frac{\ell}{N})}{2})H(x-\frac{k}{N}, y-\frac{\ell}{N})\\ &\quad(H(x-\frac{k}{N}, y-\frac{\ell}{N})-1)|\frac{i-j}{N}|^{H(x-\frac{k}{N}, y-\frac{\ell}{N})-2}\Big|dxdy.\end{aligned}$$

Then by Lemma 2.2.3

$$\begin{aligned}|(I)| &\le |i-j|^{2H(t)-2}N^{-2H(t)}\int_0^1\int_0^1\sum_{k=0}^{1}\sum_{\ell=0}^{1}\Big|A(\frac{H(t+\frac{x-k}{N},t+\frac{y-l}{N})}{2})H(t+\frac{x-k}{N},t+\frac{y-l}{N})\\ &\quad (H(t+\frac{x-k}{N},t+\frac{y-l}{N})-1)|\frac{i-j}{N}|^{H(t+\frac{x-k}{N},t+\frac{y-l}{N})-2H(t)}\Big|N^{-H(t+\frac{x-k}{N},t+\frac{y-l}{N})}dxdy\\ &\le C_4(\ln|i-j|)^2|i-j|^{2H(t)-2}N^{-2H(t)-2\beta}.\end{aligned}$$

Let

$$\begin{aligned}(II) &= \int_{t+\frac{i}{N}}^{t+\frac{i+1}{N}}\int_{t+\frac{j}{N}}^{t+\frac{j+1}{N}}\sum_{k=0}^{1}\sum_{\ell=0}^{1}A(\frac{H(x-\frac{k}{N},y-\frac{l}{N})}{2})\\ &\quad H(x-\frac{k}{N},y-\frac{l}{N})(H(x-\frac{k}{N},y-\frac{l}{N})-1)\\ &\quad (|\frac{i-j}{N}|^{H(x-\frac{k}{N},y-\frac{l}{N})-2}-|(x-\frac{k}{N})-(y-\frac{l}{N})|^{H(x-\frac{k}{N},y-\frac{l}{N})-2})\\ &\quad -\sum_{k=0}^{1}\sum_{\ell=0}^{1}A(H(t))(2H(t))(2H(t)-1)\\ &\quad (|\frac{i-j}{N}|^{2H(t)-2}-|(x-\frac{k}{N})-(y-\frac{l}{N})|^{2H(t)-2})dydx\end{aligned}$$

and so

$$\begin{aligned}|(II)| &\le \int_{t+\frac{i-1}{N}}^{t+\frac{i+1}{N}}\int_{t+\frac{j-1}{N}}^{t+\frac{j+1}{N}}\Big|A(\frac{H(x,y)}{2})\\ &\quad H(x,y)(H(x,y)-1)(|\frac{i-j}{N}|^{H(x,y)-2}-|x-y|^{H(x,y)-2})\\ &\quad -A(H(t))(2H(t))(2H(t)-1)\\ &\quad (|\frac{i-j}{N}|^{2H(t)-2}-|x-y|^{2H(t)-2})\Big|dydx.\end{aligned}$$

When $(x,y)\in[t+\frac{i-1}{N},t+\frac{i+1}{N}]\times[t+\frac{j-1}{N},t+\frac{j+1}{N}]$ and $|i-j|>2$ then $|i-j|-2\le N|x-y|\le |i-j|+2$ and so $|\ln\frac{|i-j|}{N}-\ln|x-y||\le 4|i-j|^{-1}$. It follows that $||\frac{i-j}{N}|^{2H(t)-2}-|x-y|^{2H(t)-2}|\le |i-j|^{2H(t)-3}N^{-2H(t)+2}$. Then

$$\begin{aligned}&\Big|A(\frac{H(x,y)}{2})H(x,y)(H(x,y)-1)(|\frac{i-j}{N}|^{H(x,y)-2}-|x-y|^{H(x,y)-2})\\ &-A(H(t))(2H(t))(2H(t)-1)(|\frac{i-j}{N}|^{2H(t)-2}-|x-y|^{2H(t)-2})\Big|\\ \le\ &|A(\frac{H(x,y)}{2})H(x,y)(H(x,y)-1)-A(H(t))(2H(t))(2H(t)-1)|\\ &\cdot||\frac{i-j}{N}|^{2H(t)-2}-|x-y|^{2H(t)-2}|\\ &+A(\frac{H(x,y)}{2})H(x,y)(H(x,y)-1)|\frac{i-j}{N}|^{2H(t)-2}\\ &\cdot\Big((1-(\frac{N|x-y|}{|i-j|})^{2H(t)-2})(1-|x-y|^{H(x,y)-2H(t)})\\ &+(|\frac{i-j}{N}|^{H(x,y)-2H(t)}-|x-y|^{H(x,y)-2H(t)})\Big)\\ \le\ &C_5(\ln|i-j|)^2|i-j|^{2H(t)-3}N^{2-2H(t)-\gamma\beta}\end{aligned}$$

by expanding out in Taylor series. Hence

$$|(II)| \leq C_5(\ln|i-j|)^2|i-j|^{2H(t)-3}N^{-2H(t)-\gamma\beta}. \tag{2.12}$$

Adding equations (2.12) and (2.12) and using Corollary 2.2.7 proves equation (2.11) when $|i-j|>2$. The case when $|i-j|\leq 2$ is proved similarly to equation (2.10). $\square$

2.4.2 Corollary. There exist constants $C_7, C_8 > 0$ such that for all N

$$C_7N^{1-\gamma-2H(t)} \leq EV_N(t) \leq C_8N^{1-\gamma-2H(t)}.$$

2.4.3 Lemma. For each $0<\epsilon<\beta$ there exists a constant $C_9>0$ such that for all $N>1$ when $(1-\gamma)(4H(t)-3)-4\beta<0$

$$E(V_N(t)-EV_N(t))^2 \leq C_9N^{1-\gamma-4H(t)}$$

and otherwise

$$E(V_N(t)-EV_N(t))^2 \leq C_9(\ln N)^4N^{1-\gamma-4H(t)+(1-\gamma)(4H(t)-3)-4\beta}.$$

Proof. By Theorem 3.9 of [Janson, 1997],

$$\begin{aligned}
E(V_N(t)-EV_N(t))^2 &= E(\sum_{i=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor}((X_{i+1}-X_i)^2-E(X_{i+1}-X_i)^2))^2 \\
&= 2\sum_{i=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor}\sum_{j=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor}(E(X_{i+1}-X_i)(X_{j+1}-X_j))^2.
\end{aligned}$$

If $(1-\gamma)(4H(t)-3)-4\beta<0$ then by Lemma 2.4.1,

$$\begin{aligned}
&\sum_{j=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor+1}(E(X_{i+1}-X_i)(X_{j+1}-X_j))^2 \\
\leq\ & (4-4^{-H})^2A(H(t))^2N^{-4H(t)}+C_3{}^2(\ln N)^2N^{-4H(t)-2\gamma\beta} \\
&+12N^{-4H(t)}\sum_{j=1}^{\lfloor N^{1-\gamma}\rfloor}C_4|i-j|^{4H(t)-4}(\ln(N))^4N^{-2H(t)-4\beta} \\
&+C_5|i-j|^{4H(t)-6}(\ln(N))^4N^{-2\gamma\beta}+C_6|i-j|^{4H(t)-8} \\
\leq\ & c_1N^{-4H(t)}
\end{aligned}$$

and then

$$\begin{aligned}
E(V_N(t)-EV_N(t))^2 &\leq 2\sum_{i=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor+1}c_1N^{-4H(t)} \\
&\leq C_9N^{1-\gamma-4H(t)}.
\end{aligned}$$

Otherwise

$$
\begin{aligned}
&\sum_{j=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor+1} (E(X_{i+1}-X_i)(X_{j+1}-X_j))^2 \\
\leq\ & (4-4^{-H})^2 A(H(t))^2 N^{-4H(t)} + {C_3}^2(\ln N)^2 N^{-4H(t)-2\gamma\beta} \\
&+12N^{-4H(t)} \sum_{j=1}^{\lfloor N^{1-\gamma}\rfloor} C_4|i-j|^{4H(t)-4}(\ln(N))^4 N^{-2H(t)-4\beta} \\
&+C_5|i-j|^{4H(t)-6}(\ln(N))^4 N^{-2\gamma\beta} + C_6|i-j|^{4H(t)-8} \\
\leq\ & c_2(\ln N)^4 N^{-4H(t)+(1-\gamma)(4H(t)-3)-4\beta}
\end{aligned}
$$

and

$$
\begin{aligned}
E(V_N(t)-EV_N(t))^2 &\leq 2\sum_{i=-\lfloor N^{1-\gamma}\rfloor}^{\lfloor N^{1-\gamma}\rfloor+1} c_2(\ln N)^4 N^{-4H(t)+(1-\gamma)(4H(t)-3)-4\beta} \\
&\leq C_9(\ln N)^4 N^{1-\gamma-4H(t)+(1-\gamma)(4H(t)-3)-4\beta}
\end{aligned}
$$

as required. □

2.4.4 Lemma. Almost surely,

$$\lim_{N\to\infty} \frac{V_N(t)}{EV_N(t)} = 1.$$

Proof. This result is an application of the Borel-Cantelli Lemma. Let $\epsilon > 0$. By Lemma 2.4.3 there exists a δ such that $\|V_N(t) - EV_N(t)\|_2 \leq cN^{1-\gamma-4H(t)+\delta}$ and $\frac{1}{2}(1-\gamma+\delta) > 0$. Then by Lemma 2.4.2,

$$
\begin{aligned}
P\left(\left|\frac{V_N(t)}{EV_N(t)} - 1\right| > \epsilon\right) &= P\left(|V_N(t) - EV_N(t)| > \epsilon EV_N\right) \\
&\leq P\left(|V_N(t) - EV_N(t)| > \epsilon\frac{C_7}{\sqrt{c}} N^{\frac{1}{2}(1-\gamma+\delta)}\|V_N(t) - EV_N(t)\|_2\right).
\end{aligned}
$$

Since $V_N(t) - EV_N(t)$ is a quadratic polynomial of Gaussian random variables, by Theorem 6.7 of [Janson, 1997], when $\epsilon\frac{C_7}{\sqrt{c}}N^{\frac{1}{2}(1-\gamma+\delta)} > 2$,

$$P\left(|V_N(t) - EV_N(t)| > \epsilon\frac{C_7}{\sqrt{c}} N^{\frac{1}{2}(1-\gamma+\delta)}\|V_N(t) - EV_N(t)\|_2\right) \leq \exp\left(-\kappa\epsilon\frac{C_7}{\sqrt{c}} N^{\frac{1}{2}(1-\gamma+\delta)}\right)$$

where $\kappa > 0$ is an absolute constant. This implies that

$$\sum_{N=1}^{\infty} P\left(\left|\frac{V_N(t)}{EV_N(t)} - 1\right| > \epsilon\right) < \infty$$

and so the result follows from the Borel-Cantelli Lemma. □

2.4.5 Theorem. The estimator $\check{H}_N(t)$ converges almost surely to $H(t)$ as $N \to \infty$.

Proof. By Lemma 2.4.1,

$$\lim_{N\to\infty} \log_2 \frac{EV_{N/2}(t)}{EV_N(t)} = -(1-\gamma) + 2H(t)$$

and so by Lemma 2.4.4

$$\lim_{N\to\infty}\log_2\frac{V_{N/2}(t)}{V_N(t)}=\lim_{N\to\infty}\log_2\frac{EV_{N/2}(t)}{EV_N(t)}+\log_2\frac{V_{N/2}(t)}{EV_{N/2}(t)}+\log_2\frac{EV_N(t)}{V_N(t)}=-(1-\gamma)+2H(t)$$

almost surely. Hence

$$\lim_{N\to\infty}\frac{1}{2}\left(\log_2\frac{1+2\lfloor N^{1-\gamma}\rfloor}{1+2\lfloor (N/2)^{1-\gamma}\rfloor}+\log_2\frac{V_{N/2}(t)}{V_N(t)}\right)=H(t)$$

almost surely. □

2.4.6 Lemma. When $(1-\gamma)(4H(t)-3)-4\beta<0$

$$N^{-(1-\gamma)}E\left(\frac{V_N-EV_N}{EV_N}-\frac{V_{N/2}-EV_{N/2}}{EV_{N/2}}\right)^2\to C_{10}$$

where

$$\begin{aligned}C_{10} &= \frac{(1+2^{-(1-\gamma)})}{4(4-4^H(t))^2}\sum_{i=-\infty}^{\infty}\big(-|i+2|^{2H(t)}+4|i+1|^{2H(t)}-6|i|^{2H(t)}\\ &\quad +4|i-1|^{2H(t)}-|i-2|^{2H(t)}\big)^2+\frac{2^{1-\gamma}}{4(4-4^H(t))^2}\sum_{i=-\infty}^{\infty}\big(-|i+3|^{2H(t)}+2|i+2|^{2H(t)}\\ &\quad +|i+1|^{2H(t)}-4|i|^{2H(t)}+|i-1|^{2H(t)}+2|i-2|^{2H(t)}-|i+3|^{2H(t)}\big)^2\end{aligned}$$

This follows by Lemma 2.4.1.

2.4.7 Lemma. There exists a constant $C_{11}>0$ such that for all $N>1$,

$$\left|2H(t)-\log_2\frac{1+2\lfloor N^{1-\gamma}\rfloor}{1+2\lfloor (N/2)^{1-\gamma}\rfloor}-\log_2\frac{EV_{N/2}}{EV_N}\right|\le C_{11}\ln(N)N^{-\gamma\beta}.$$

Proof. By equation (2.10)

$$\begin{aligned}&2H(t)-\log_2\frac{1+2\lfloor N^{1-\gamma}\rfloor}{1+2\lfloor (N/2)^{1-\gamma}\rfloor}-\log_2\frac{EV_{N/2}}{EV_N}\\ \le\ & 2H(t)-\log_2((4-4^{H(t)})A(H(t))(N/2)^{-2H(t)}-C_3(\ln(N/2))(N/2)^{-2H(t)-\gamma\beta})\\ &+\log_2((4-4^{H(t)})A(H(t))N^{-2H(t)}+C_3(\ln N)N^{-2H(t)-\gamma\beta})\\ =\ & -\log_2((4-4^{H(t)})A(H(t))-C_3(\ln(N/2))(N/2)^{-\gamma\beta})\\ &+\log_2((4-4^{H(t)})A(H(t))+C_3(\ln N)N^{-\gamma\beta})\\ &\le C_{11}\ln(N)N^{-\gamma\beta}.\end{aligned}$$

The lower bound holds similarly proving the result. □

2.4.8 Lemma. If $(1-\gamma)(4H(t)-3)-4\beta<0$ then

$$\frac{1}{\sqrt{C_{10}N^{-(1-\gamma)}}}\left(\frac{V_N-EV_N}{EV_N}-\frac{V_{N/2}-EV_{N/2}}{EV_{N/2}}\right)\xrightarrow{d}N(0,1)$$

as $N\to\infty$.

Proof. Let H_N be the Gaussian Hilbert space generated by $\{X_{i+1} - X_i : -\lfloor N^{1-\gamma} \rfloor \leq i \leq \lfloor N^{1-\gamma} \rfloor\}$. Then $\frac{V_N - EV_N}{EV_N} - \frac{V_{N/2} - EV_{N/2}}{EV_{N/2}} \in H_N^{:2:}$ and so we will apply the representation formula from Theorem 6.1 in [Janson, 1997]. Let $\widetilde{T}_N : H_N \mapsto H_N$ be the operator $\widetilde{T}_N(\xi) = \frac{1}{2}\pi_1((V_N - EV_N)\xi)$ where π_1 is the orthogonal projection onto H_N. Acting on vectors of the form $(s_i(X_{i+1} - X_i))_{-\lfloor N^{1-\gamma} \rfloor \leq i \leq \lfloor N^{1-\gamma} \rfloor}$, $\widetilde{T}_N$ is the matrix

$$([E(X_{i+1} - X_i)(X_{j+1} - X_j)])_{i,j=-\lfloor N^{1-\gamma} \rfloor \ldots \lfloor N^{1-\gamma} \rfloor}.$$

By a standard result in linear algebra the largest eigenvalue of $\widetilde{T}_N$ has absolute value at most

$$\max_i \sum_{j=-\lfloor N^{1-\gamma} \rfloor}^{\lfloor N^{1-\gamma} \rfloor} |E(X_{i+1} - X_i)(X_{j+1} - X_j)|.$$

By Lemma 2.4.1

$$\begin{aligned} & \sum_{j=-\lfloor N^{1-\gamma} \rfloor}^{\lfloor N^{1-\gamma} \rfloor} |E(X_{i+1} - X_i)(X_{j+1} - X_j)| \\ \leq\ & C_4 |i-j|^{2H(t)-2} (\ln(N))^2 N^{-2H(t)-2\beta} \\ & + C_5 |i-j|^{2H(t)-3} (\ln(N))^2 N^{-2H(t)-\gamma\beta} + C_6 |i-j|^{2H(t)-4} N^{-2H(t)}. \end{aligned}$$

Then when $(2H(t) - 1)(1 - \gamma) < 2\beta$

$$\sum_{j=-\lfloor N^{1-\gamma} \rfloor}^{\lfloor N^{1-\gamma} \rfloor} |E(X_{i+1} - X_i)(X_{j+1} - X_j)| \leq c_1 N^{-2H(t)}.$$

Otherwise when $(2H(t) - 1)(1 - \gamma) \geq 2\beta$

$$\sum_{j=-\lfloor N^{1-\gamma} \rfloor}^{\lfloor N^{1-\gamma} \rfloor} |E(X_{i+1} - X_i)(X_{j+1} - X_j)| \leq c_2 (\ln(N))^2 N^{-2H(t)+(2H(t)-1)(1-\gamma)-2\beta}.$$

Now let $\widetilde{T}'_N$ be the operator

$$\widetilde{T}'_N(\xi) = \frac{1}{2}\pi_1 \left(\frac{1}{\sqrt{C_{10} N^{-(1-\gamma)}}} \left(\frac{V_N - EV_N}{EV_N} - \frac{V_{N/2} - EV_{N/2}}{EV_{N/2}} \right) \xi \right).$$

Then by Theorem 6.1 in [Janson, 1997]

$$\frac{1}{\sqrt{C_{10} N^{-(1-\gamma)}}} \left(\frac{V_N - EV_N}{EV_N} - \frac{V_{N/2} - EV_{N/2}}{EV_{N/2}} \right)$$

can be rewritten as

$$\sum_j \lambda_{j,N} (\xi_{j,N}^2 - 1)$$

where the $\lambda_{j,N}$ are the eigenvalues of $\widetilde{T}'_N$ and the $\xi_{j,N}$ are independent (for fixed N) $N(0,1)$ distributed random variables. Since

$$\widetilde{T}'_N = \frac{1}{\sqrt{C_{10} N^{-(1-\gamma)}}} \left(\frac{\widetilde{T}_N}{EV_N} - \frac{\widetilde{T}_{N/2}}{EV_{N/2}} \right)$$

the maximum eigenvalue of $\widetilde{T}'_N$ is at most

$$\frac{1}{\sqrt{C_{10}N^{-(1-\gamma)}}}\left(\frac{c_1 N^{-2H(t)}}{C_7 N^{1-\gamma-2H(t)}}+\frac{c_1(N/2)^{-2H(t)}}{C_7(N/2)^{1-\gamma-2H(t)}}\right)\leq c_3 N^{-(1-\gamma)/2}$$

when $(2H(t)-1)(1-\gamma)<2\beta$. When $(2H(t)-1)(1-\gamma)\geq 2\beta$ the maximum eigenvalue of $\widetilde{T}'_N$ is at most

$$\frac{1}{\sqrt{C_{10}N^{-(1-\gamma)}}}\left(\frac{c_2(\ln(N))^2 N^{-2H(t)+(2H(t)-1)(1-\gamma)-2\beta}}{C_7 N^{1-\gamma-2H(t)}}\right.$$
$$\left.+\frac{c_2(\ln(N))^2(N/2)^{-2H(t)+(2H(t)-1)(1-\gamma)-2\beta}}{C_7(N/2)^{1-\gamma-2H(t)}}\right)\leq c_3(\ln(N))^2 N^{-(1-\gamma)/2+(2H(t)-1)(1-\gamma)-2\beta}.$$

In either case $\max_j|\lambda_{j,N}|\to 0$ as $N\to\infty$. By Theorem 7.1.2 of [Chung, 2001]

$$\frac{1}{\sqrt{C_{10}N^{-(1-\gamma)}}}\left(\frac{V_N-EV_N}{EV_N}-\frac{V_{N/2}-EV_{N/2}}{EV_{N/2}}\right)=\sum_j\lambda_{j,N}(\xi^2_{j,N}-1)\xrightarrow{d}N(0,1)$$

as $N\to\infty$. □

2.4.9 Theorem. [Central Limit Theorem] If $(1-\gamma)(4H(t)-3)-4\beta<0$ and

$$\gamma>\frac{1}{1+2\beta}.$$

then

$$\frac{\ln 2}{\sqrt{C_{10}N^{-(1-\gamma)}}}\left(H(t)-\check{H}_N(t)\right)\xrightarrow{d}N(0,1)$$

as $N\to\infty$.

Proof. By Lemma 2.4.7,

$$\left|H(t)-\frac{1}{2}\left(\log_2\frac{1+2\lfloor N^{1-\gamma}\rfloor}{1+2\lfloor (N/2)^{1-\gamma}\rfloor}+\log_2\frac{EV_{N/2}}{EV_N}\right)\right|\leq C_{11}\ln(N)N^{-\gamma\beta}$$

and so

$$\frac{\ln 2}{\sqrt{C_{10}N^{-(1-\gamma)}}}\left(H(t)-\frac{1}{2}\left(\log_2\frac{1+2\lfloor N^{1-\gamma}\rfloor}{1+2\lfloor (N/2)^{1-\gamma}\rfloor}+\log_2\frac{EV_{N/2}}{EV_N}\right)\right)\to 0.$$

Then it is sufficient to prove that

$$\frac{\ln 2}{\sqrt{C_{10}N^{-(1-\gamma)}}}\left(\log_2\frac{V_N}{EV_N}-\log_2\frac{V_{N/2}}{EV_{N/2}}\right)\xrightarrow{d}N(0,1).$$

This follows from Lemma 2.4.8 since $\frac{V_N}{EV_N}$ converges almost surely to 1 and since $\log_2(x)=\frac{1}{\ln 2}(x-1)+o(x-1)$ completing the result. □

2.4.10 Remark. The condition that $4\beta>(4H(t)-3)(1-\gamma)$ is always satisfied when $H(t)\leq\frac{3}{4}$. For all functions $H(t)$, γ can be chosen such that $4\beta>(4H(t)-3)(1-\gamma)$.

2.5 Generalizations

There are numerous generalizations of MBM. As mentioned earlier some have been directed towards expanding the class of functions $H(t)$ for which it can be defined. Other have generalized MBM by replacing the Wiener noise term with another Lévy measure. These generalizations also exhibit extreme unwanted changes in magnitude as $H(t)$ changes. We briefly propose redefinitions along the lines of IFWN.

[Lacaux, 2004] defined Real Harmonizable Fractional Lévy motion as

$$M_H(t) = \mathrm{Re} \int_{\mathbb{R}} \left(\frac{e^{i\xi t} - 1}{|\xi|^{H(t)+1/2}} \right) \widetilde{L}(d\xi)$$

where $\widetilde{L}$ is a complex Lévy measure having moments of all orders. Following the approach of this chapter it can be redefined as

$$Y_{H(t)}(t) = \mathrm{Re} \int_{\mathbb{R}} \left(\int_0^t \frac{i\xi e^{i\xi s}}{|\xi|^{H(s)+1/2}} ds \right) \widetilde{L}(d\xi). \tag{2.13}$$

[Stoev and Taqqu, 2004b] developed a multifractional stable motion using the moving average representation of linear fractional stable motion. We would define multifractional stable motion as

$$Y_{H(t)}(t) = \int_{\mathbb{R}} \left(\int_0^t \frac{i\xi e^{i\xi s}}{|\xi|^{H(s)+1-1/\alpha}} ds \right) \widetilde{M}(d\xi).$$

where $\widetilde{M}$ is a complex isotropic $S\alpha S$ random measure (see [Samorodnitsky and Taqqu, 1994], Definition 2.6.3) and $H(t)$ satisfies conditions conditions (2.4) and (2.5). When $H(t)$ is constant this reduces to harmonizable fractional stable motion as in equation (7.7.1) of [Samorodnitsky and Taqqu, 1994].

Chapter 3

Multifractal and Monofractal Subordinator models in Finance

This chapter compares the evidence for two models for the log returns of financial data. These returns processes, which may come from equities, indices, foreign exchange or commodities, exhibit complex scaling behavior. Of course the standard model is Brownian motion which is $\frac{1}{2}$-self-similar. However, it is apparent from real data that the returns exhibit long memory and tails heavier than the normal distribution.

Various authors including [Peters, 1991], [Bouchaud and Sornette, 1994] and [Elliott and van der Hoek, 2003] have built models where the returns are modelled as fractional Brownian motion with $H > \frac{1}{2}$ as a way of incorporating long-memory into the model. As fractional Brownian motion is a non-semimartingale these models have led to the development of new forms of stochastic calculus which extend beyond the traditional Itô calculus for semimartingales. However, [Rogers, 1997] showed that arbitrage opportunities exist when the returns are non-semimartingales. While the model in [Elliott and van der Hoek, 2003], which uses Wick calculus, is arbitrage free [Bender and Elliott, 2004] noted that this is because its class of admissible portfolios is "rather odd".

More importantly fractional Brownian motion has correlated increments which are long range dependent. Sample autocorrelations of real log returns, however, die away rapidly and are statistically insignificant. Persistence in autocorrelations of the absolute value of returns is very strong. To illustrate this Figure 3.1 displays the sample autocorrelations of the returns and absolute values of the returns for 11413 observations of the S & P 500 from January 1960 to May 2005. The absolute values of the returns show positive autocorrelations which are significant out to a lag of 1000 days.

Another important property found in financial data is heavy tails. The S & P returns, when normalized to have mean 0 and standard deviation 1, contain values of -24.4, 9.4, -9.3 and -7.7 which are inconceivably large for normally distributed random variables. [Heyde and Kou, 2004] noted that there is considerable debate about the distribution of financial returns. [Barndorff-Nielsen and Shephard, 2001] argued for the use of exponentially tailed distributions. On the other hand [Mandelbrot, 1963] found that cotton prices

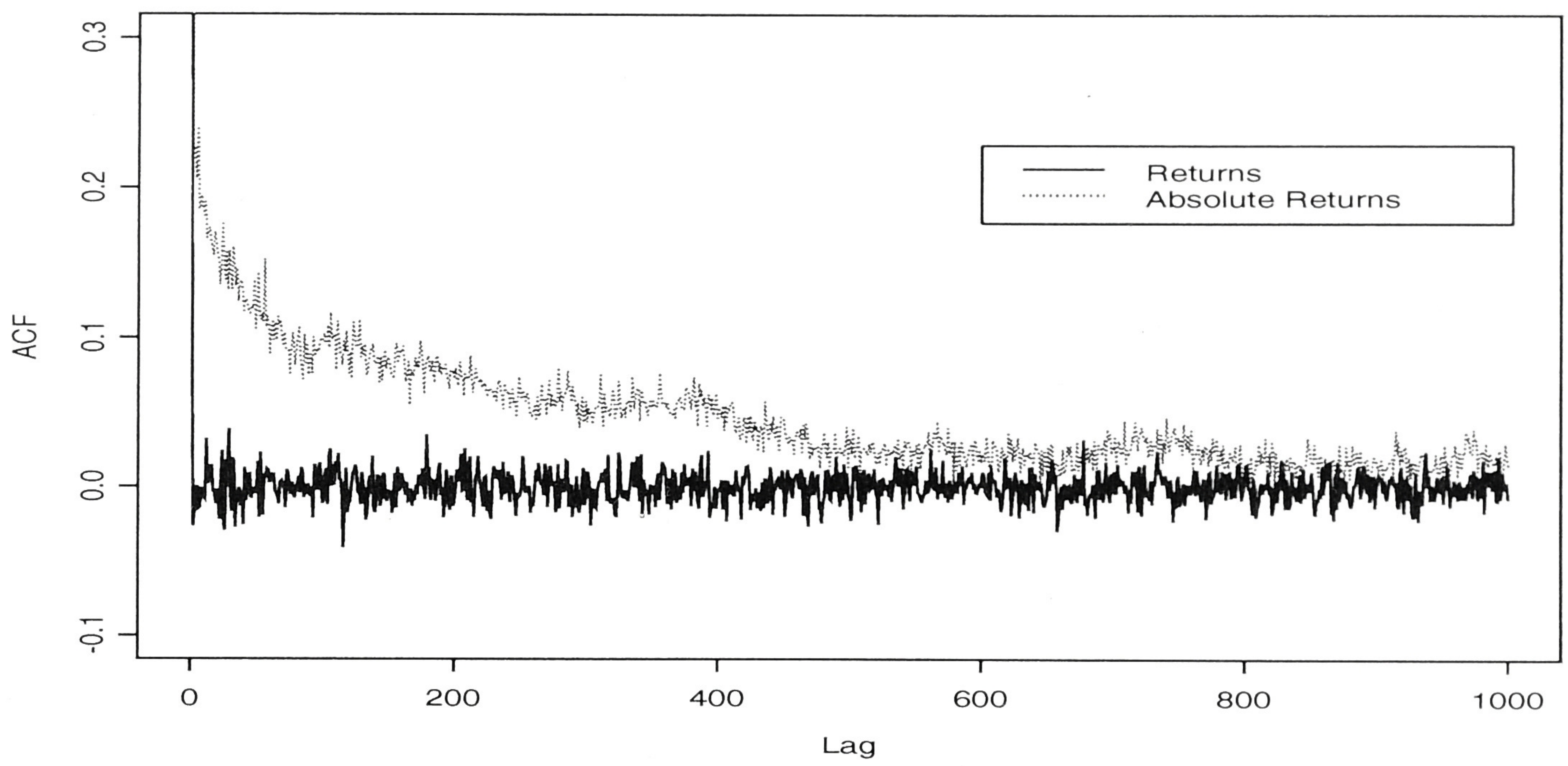

Figure 3.1: Autocorrelations of S & P 500 Returns and Absolute Returns

could be fitted with stable distributions with infinite variance. [Heyde and Liu, 2001], citing [Hurst et al., 1997] and others, advocated a Student t distribution with ν degrees of freedom for $3 \leq \nu \leq 6$ which has tails of Pareto type

$$P(|X| > x) \sim cx^{-\nu}. \tag{3.1}$$

A standard approach to modelling uncorrelated increments which can also incorporate heavy tails is a subordinator model. The returns are modelled as

$$X_t = \mu + \sigma(W(T_t) - W(T_{t-1})) \stackrel{d}{=} \mu + \sigma(T_t - T_{t-1})^{\frac{1}{2}} W(1). \tag{3.2}$$

Here T_t is an increasing stochastic process with stationary increments, independent of the noise process $W(t)$, which models the underlying market activity time rather than the "clock time". We assume that $ET_t = t$ and so as t goes to infinity it follows from the ergodic theorem that $\frac{1}{t}T_t \to 1$ almost surely. For a discussion of subordinator models see [Rachev and Mittnik, 2000]. [Heyde, 1999] showed that

$$\begin{aligned} \operatorname{cov}(X_t, X_{t+k}) &= 0 \\ \operatorname{cov}(|X_t|, |X_{t+k}|) &= \frac{2\sigma^2}{\pi} \operatorname{cov}((T_t - T_{t-1})^{\frac{1}{2}}, (T_{t+k} - T_{t+k-1})^{\frac{1}{2}}). \end{aligned}$$

Hence while the $\{X_t\}$ are always uncorrelated the $\{|X_t|\}$ are long range dependent if $\{T_t - T_{t-1}\}$ are long range dependent.

The rest of this chapter focuses on two models, one multifractal and the other monofractal and examines the respective evidence.

3.1 Multifractal Stochastic Processes

Multifractals were introduced in [Mandelbrot, 1972] as measures to model turbulence. The concept was extended in [Mandelbrot et al., 1997] to stochastic processes as a generalization of self-similar stochastic processes. The definition of a multifractal is motivated by generalizing the scaling rule for self-similar processes to

$$X(ct) \stackrel{d}{=} M(c)X(t) \tag{3.3}$$

for $0 < c < 1$ where $M(c)$ is a random variable independent of $X(t)$ and equality is in finite dimensional distributions. When $M(c) = c^H$ this is the definition for a self-similar process. The actual definition of a multifractal process, as given in [Mandelbrot et al., 1997], is defined in terms of the moments of the process and includes processes satisfying equation (3.3).

3.1.1 Definition. A stochastic process $X(t)$ is multifractal if it has stationary increments and there exist functions $c(q)$ and $\tau(q)$ and positive constants $\mathcal{Q}$ and $\mathcal{T}$ such that

$$\forall q \in [0, \mathcal{Q}], \forall t \in [0, \mathcal{T}] \quad E|X(t)|^q = c(q)t^{\tau(q)}. \tag{3.4}$$

The function $\tau(q)$ is called the scaling function.

While this definition is the standard definition of a multifractal process most processes studied as multifractals only obey it for particular values of t or sometimes for asymptotically small t. The condition of stationary increments is also quite often relaxed.

It follows from the definition that

$$\ln E|X(t)|^q = \ln c(q) + \tau(q) \ln t \tag{3.5}$$

and so $X(t)$ is multifractal if for each q, $\ln E|X(t)|^q$ scales linearly with $\ln t$ and the slope is $\tau(q)$. This is the primary test used to determine if a process is multifractal.

An H-self-similar process is multifractal with $\tau = Hq$. [Mandelbrot et al., 1997] showed that the scaling function is concave for all multifractals with the following argument. Let w_1, w_2 be positive weights with $w_1 + w_2 = 1$ and let $0 \leq q_1, q_2 \leq \mathcal{Q}$ and $q = q_1 w_1 + q_2 w_2$. Then by Hölder's inequality

$$E|X(t)|^q \leq [E|X(t)|^{q_1}]^{w_1} [E|X(t)|^{q_2}]^{w_2}$$

and so

$$\ln c(q) + \tau(q) \ln t \leq [w_1 \tau(q_1) + w_2 \tau(q_2)] \ln t + [w_1 \ln c(q_1) + w_2 \ln c(q_2)]. \tag{3.6}$$

Letting t go to zero we have $\tau(q) \geq w_1 \tau(q_1) + w_2 \tau(q_2)$ so τ is concave. If $\mathcal{T} = \infty$ we can let t go to ∞ and we get the reverse the inequality $\tau(q) \leq w_1 \tau(q_1) + w_2 \tau(q_2)$. It follows that $\mathcal{T} = \infty$ implies that τ is linear and so $X(t)$ is self-similar.

So multifractals which are not self-similar can only be defined on finite intervals. [Mandelbrot et al., 1997] asserts that this "has little consequence for financial modelling, since multifractal processes can be defined on arbitrarily large time intervals" but this is consequential when the distribution of $X(1)$ is known or can be estimated from the data.

Suppose that τ is concave and non-linear. Then $\tau(q_1)/q_1 > \tau(q_2)/q_2$ for some $q_1 < q_2$. Since $(E|X(t)|^{q_1})^{\frac{1}{q_1}} \leq (E|X(t)|^{q_2})^{\frac{1}{q_2}}$ we have

$$(c(q_1)t^{\tau(q_1)})^{\frac{1}{q_1}} \leq (c(q_2)t^{\tau(q_2)})^{\frac{1}{q_2}}. \tag{3.7}$$

Now $c(q) = E|X(1)|^q$ so $c(q)$ is determined by the distribution of $X(1)$. Hence if τ and the distribution of $X(1)$ are known rearranging equation (3.7) yields

$$t \leq \left[\frac{c(q_2)^{\frac{1}{q_2}}}{c(q_1)^{\frac{1}{q_1}}}\right]^{(\tau(q_1)/q_1 - \tau(q_2)/q_2)^{-1}} \tag{3.8}$$

which is a fixed upper bound for t. When these parameters were estimated for the S & P 500 returns assuming $\mathcal{Q} \geq 3$, setting $q_1 = 2.9$ and $q_2 = 3.0$, equation (3.8) gave an upper bound of 1604 days. Estimates from the Dow Jones components gave upper bounds ranging from 171 days to over 10^{41} days. In some contexts these upper bounds may be a serious shortcoming for the model. These estimates are very sensitive to τ which is in turn very sensitive to the extreme data points as is discussed later in the chapter.

An important associated concept is the multifractal spectrum. It is the Lengedre transform of $\tau(q) - 1$ and is given by

$$f(\alpha) = \inf_q \{q\alpha + 1 - \tau(q)\} \tag{3.9}$$

where it is defined. For self-similar processes it is only defined at H with $f(H) = 1$. The multifractal spectrum plays an important role in multifractal measures where it represents the fractal dimensions of sets where the measure has certain limiting intensities. The analogous definition for multifractal processes is the dimension of sets with local Hölder exponent α (see [Calvet et al., 1997] for details). However, for multifractal processes the multifractal spectrum is only used as a tool for fitting the model to data.

The motivating example of a multifractal process is the cascade. They were first introduced as measures in [Mandelbrot, 1974] and can be defined on the interval $[0, 1]$ as follows. Define a sequence of random measures μ_n by

$$\mu_n(dt) = \prod_{i=1}^{n} M_{\eta_1,\eta_2,\ldots,\eta_i}(dt) \tag{3.10}$$

where t has expansion $t = 0.\eta_1\eta_2\ldots$ in base b and the $M_{\eta_1,\eta_2,\ldots,\eta_i}$ are a collection of positive iid random variables with distribution M where $EM = 1$. [Kahane and Peyrière, 1976] showed that the almost sure vague limit of μ_n exists, denoted as μ. The stochastic process $X(t)$ is defined as $X(t) = \mu([0, t])$. It is easy to check that equation (3.4) holds when $t = b^{-n}$. Of course $X(t)$ does not fully satisfy the definition of a multifractal as equation (3.4) does not hold except when t is of the form b^{-n} and $X(t)$ does not even have stationary increments. Even though cascades do not satisfy the formal definition they remain the prototype model for multifractal processes. Most processes studied as multifractals only satisfy the strict definition asymptotically as t goes to 0 but a notable exception is a class of multifractal processes given in [Bacry and Muzy, 2003] which satisfy the definition as well as equation (3.3) although this does not appear to add to its utility as a model.

Multifractals overcome an important limitation of self-similar stochastic processes which is they can be positive and still have finite mean as in the case of cascades. When $X(t)$ is positive and $EX(1) < \infty$ equation (3.4) implies that $\tau(1) = 1$.

3.2 Brownian motion in Multifractal Time

To address the empirical features noted at the beginning of this chapter [Mandelbrot et al., 1997] introduced a multifractal model of asset returns (MMAR) which Mandelbrot now refers to as Brownian motion in multifractal time (BMMT).

3.2.1 Definition. Brownian motion in multifractal time is defined as

$$X(t) = B_H(\theta(t)) \tag{3.11}$$

where $\theta(t)$ is a positive multifractal stochastic process and B_H is an independent fractional Brownian motion with Hurst parameter H.

The process $\theta(t)$ can be thought of as the activity time or volatility of the process. Since

$$\begin{aligned} E|B_H(\theta(t))|^q &= E|B_H(1)|^q E\theta(t)^{Hq} \\ &= c_\theta(q)t^{\tau_\theta(Hq)}E|B_H(1)|^q \end{aligned}$$

the scaling function for $X(t)$ is given by

$$\tau_X(q) = \tau_\theta(Hq) \tag{3.12}$$

and it follows that $X(t)$ is also multifractal. Since $\tau_\theta(1) = 1$ it follows that

$$\tau(\frac{1}{H}) = 1 \tag{3.13}$$

which is used to estimate H. When $H = \frac{1}{2}$ we have that $X(t)$ is a martingale and so defines an arbitrage free market. However, when $H < \frac{1}{2}$ the increments are negatively correlated and when $H > \frac{1}{2}$ the increments are positively correlated and long range dependent.

[Fisher et al., 1997] fitted a BMMT model to the log-returns of the US Dollar-Deutschmark exchange rate. This is done by fitting the model's scaling function to the estimated scaling function. Now $\widehat{\tau}$ is estimated from

$$\ln S_q(n,s) = \ln c(q) + \tau(q)\ln t \tag{3.14}$$

by ordinary least squares for a range of s where

$$S_q(n,t) = \frac{1}{\lfloor n/t \rfloor}\sum_{i=1}^{\lfloor n/s \rfloor}\left|\sum_{j=1}^{t} X_{s(i-1)+j}\right|^q .$$

Using equation (3.13) $\widehat{H}$ is estimated as 0.53. The multifractal spectrum is estimated by

$$\widehat{f}_X(\alpha) = \inf_q q\alpha + 1 - \widehat{\tau}(q). \tag{3.15}$$

The activity time $\theta(t)$ is modelled as a cascade with M having the lognormal distribution

$$-\log_b M \stackrel{d}{=} N(\lambda - 1, \sigma^2)$$

where

$$\frac{2(\lambda-1)}{\sigma^2} = \ln b$$

to ensure that $EM = 1$. Then from [Calvet et al., 1997] the multifractal spectrum is given by

$$f_X(\alpha) = 1 - \frac{(\alpha - H\lambda)^2}{4H^2(\lambda-1)}.$$

The spectrum has a maximum when $\alpha = H\lambda$ so $\widehat{\lambda}$ is estimated by setting $\widehat{H}\widehat{\lambda}$ to be the value of $\widehat{\alpha}$ which maximizes $\widehat{f}_X(\alpha)$. This gives $\widehat{\lambda} = 1.11$. A selection of Monte Carlo simulations can be seen in [Fisher et al., 1997].

[Mandelbrot, 2001a] and [Mandelbrot, 2001b] advocated a different BMMT model. The multifractal time $\theta(t)$ is a multifractal product of cylindrical pulses (MPCP) which is a continuous extension to grid-based cascades. Its construction was originally set out in [Barral and Mandelbrot, 2002] but is done equivalently in [Bacry and Muzy, 2003] as follows. Let M be a positive integrable random variable. Let N be an independently scattered compound Poisson measure on $\mathbb{R} \times (0, 1]$ whose jumps have distribution $\ln M$ with intensity

$$\Lambda(dtd\lambda) = \frac{\delta}{2\lambda^2} dtd\lambda.$$

Then with $A_{t,\epsilon} = \{(s, \lambda) | s \in [t - \lambda, t + \lambda], \lambda > \epsilon\}$ set

$$\mu_\epsilon(dt) = \epsilon^{\delta(EM-1)} \exp N(A_{t,\epsilon}). \tag{3.16}$$

The almost sure vague limit μ exists as $\epsilon \to 0$ and the MPCP process is defined as $\theta(t) = \mu([0, t])$. Unlike cascades MPCP has stationary increments. It satisfies equation (3.4) asymptotically with

$$\tau(q) = q(1 + \delta(EM - 1)) - \delta(EM^q - 1). \tag{3.17}$$

Under weak conditions on M if $\max M > 1$ then $\tau(q) \to -\infty$ as $q \to \infty$. Then $\inf\{q > 1 | \tau(q) \leq 1\}$ exists and and is the moment index of $\theta(t)$.

The advocates of BMMT as a model justify it on a number of levels. The intuitive economic understanding of multifractal activity times is described in [Calvet and Fisher, 2001] as volatility clustering on time scales ranging from "technological shocks, business and earning cycles and liquidity shocks". The fact that it can model the time periods ranging from an hour to years and that heavy tails are naturally incorporated in the model, as they emerge from the definition of MPCP, is also desirable. From an empirical perspective the main evidence for multifractal scaling is that the estimated scaling function generally appears to be concave. The next section will investigate the evidence for this.

3.3 Evidence for Multifractality

A range of estimates have been made for H in different papers based on the scaling function. As mentioned [Fisher et al., 1997] estimated $H = 0.53$ for the US/DM exchange rate while

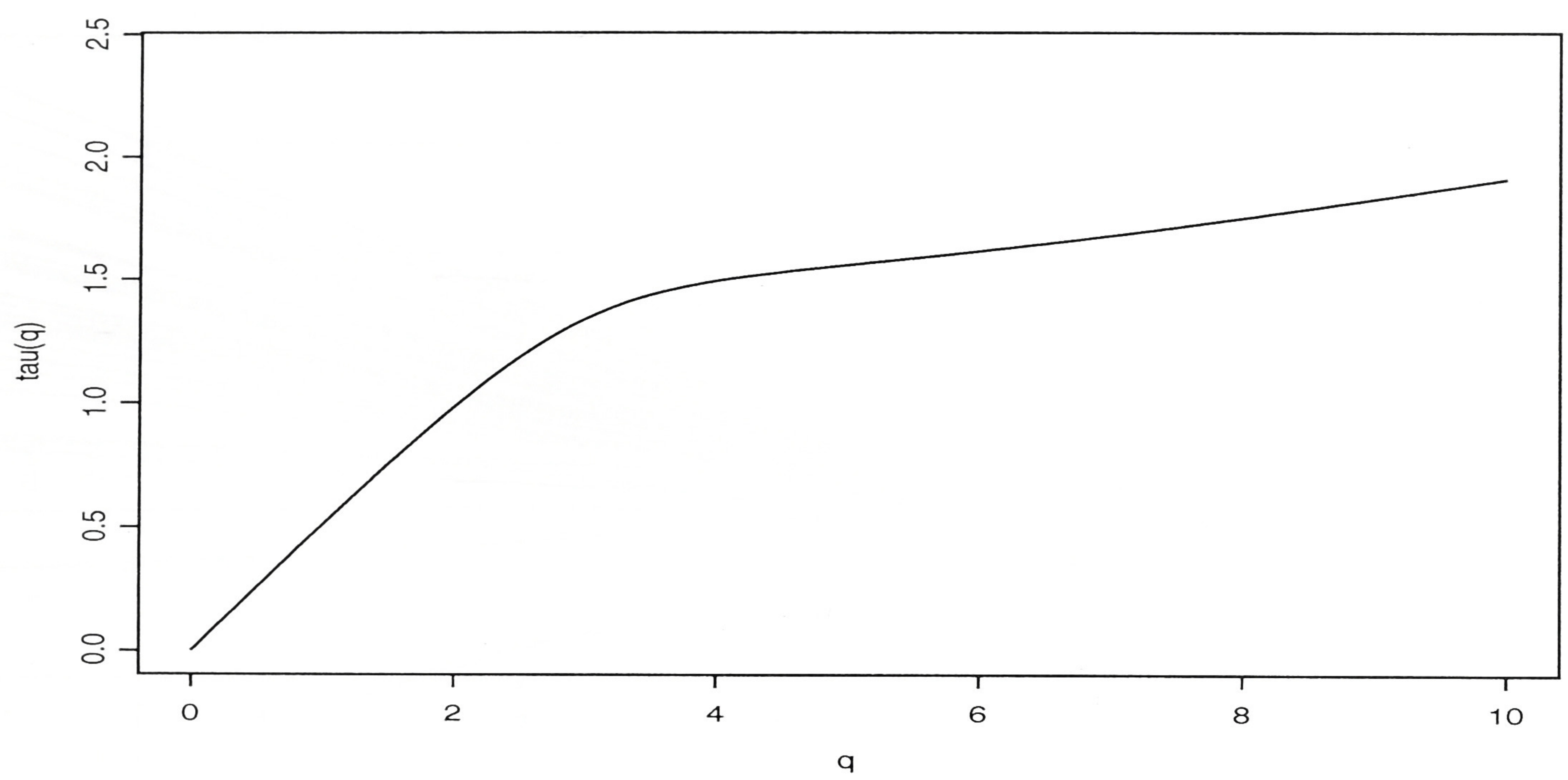

Figure 3.2: Scaling Function of S & P 500 Returns

[Jérôme, 2003] estimated $H = 0.45$ for the CAC40 index. Inspection of the S & P 500 yields an estimate of $H = 0.49$. All these values are close to 0.5 and there is so far no statistical evidence to reject the hypothesis $H = 0.5$. [Heyde, 2002] found no evidence that the signs of the returns are long range dependent so concludes that $H = 0.5$ is adequate. Fixing $H = 0.5$ has the advantage that the returns are uncorrelated and form a martingale which accords with the widely accepted Efficient Market Hypothesis.

To sensibly talk about a scaling function it is necessary for $\ln S_q(n,t)$ to vary linearly with $\ln t$. [Fisher et al., 1997] found that while this was the case for the US/DM exchange rate this was not the case with the US dollar Japanese Yen exchange rate. It was suggested that this may be linked to greater central bank intervention in the market. Investigation of a range of time series has shown that $\ln S_q(n,t)$ varies linearly with $\ln t$ for most but not all time series. Nonlinearities, when they do occur, do so in the larger moments where $\ln S_q(n,t)$ depends on a few extreme observations. The rest of this section is concerned with how the heavy tails affect the scaling function so we will "naively " calculate it without checking the assumption of linearity.

The evidence for concavity of the scaling function is much stronger. Every equity, foreign exchange or index time series checked shows concavity in the scaling function particularly for larger moments. Figure 3.2 and 3.3 show the scaling functions for the S & P 500 and the 30 components of the DJIA respectively.

While all the scaling functions are ultimately concave many of them are very linear when $q \le 2$. In fact 14 of the estimated scaling functions of the Dow components are convex at some point, 7 have $\tau(2) > 2\tau(1)$ and 3 are convex on the whole interval $[0, 2]$. It is only when q is large that concavity consistently occurs. Rather than multiscaling this can be attributed

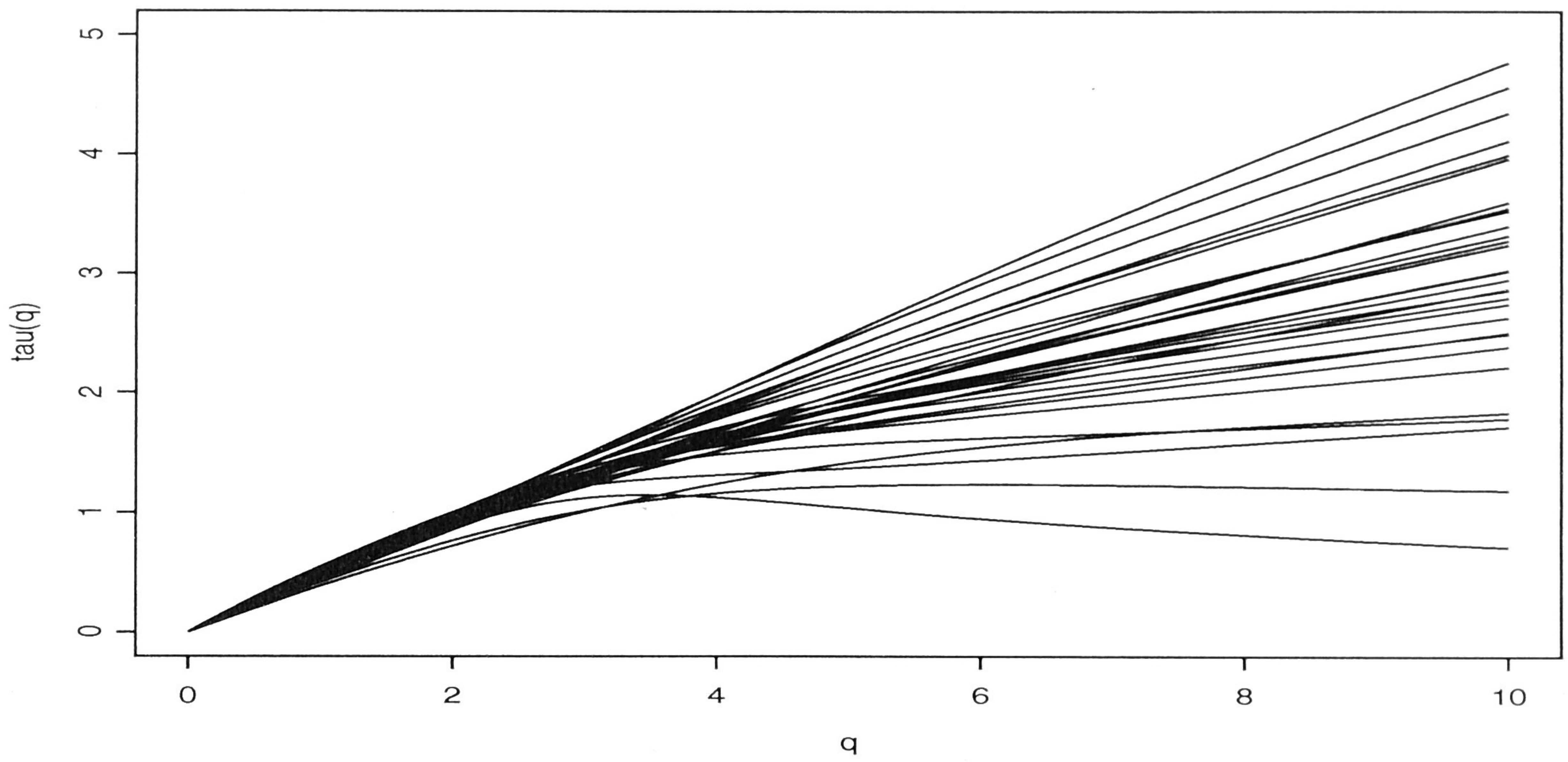

Figure 3.3: Scaling Function of DJIA Component Returns

to the heavy tails of the distribution. As noted previously with real data the moment indexed can only be estimated and particulary with dependent data these estimates may be unreliable given the sample size. For large q the estimate of $\tau(q)$ is particularly sensitive to extreme values as $\max|X_i|^q \approx \sum|X_i|^q$.

To test this we created new time series by removing extreme values from the S & P 500. The first time series had the crash of 19 October 1987 removed. The second had all events of more than 4 standard deviations removed. Figure 3.4 illustrates the effect of extreme values by recalculating scaling functions for these time series and comparing them to the original. As more extreme observations are removed τ becomes closer to $q/2$. This suggests that the concavity of τ is caused more by by the influence of extreme observations than by any multifractal scaling.

To demonstrate the effect of extreme values consider a process $Y_n = \sum_{i=1}^{n} X_n$ where the X_n are independent with t_4 distributions. As Y has zero mean and finite variance it is in the domain of attraction of the normal distribution and converges in distribution to Brownian motion. As such we should expect that for a large enough sample we should estimate $\tau = q/2$. Figure 3.5 plots $\ln S_q(n,t)$ versus $\ln t$ which shows a good degree of linearity for a typical simulation of 100000 increments.

Figure 3.6 shows τ for 20 simulations of 100000 increments. Note that $\tau(10)$ is significantly less than 5. On the other hand Figure 3.7 shows the same time series with all observations of more than 4 standard deviations removed. In this case the estimates for τ are close to the theoretical value.

The following theorem explains the effect that heavy tails have on estimating τ in a sequence with independent increments.

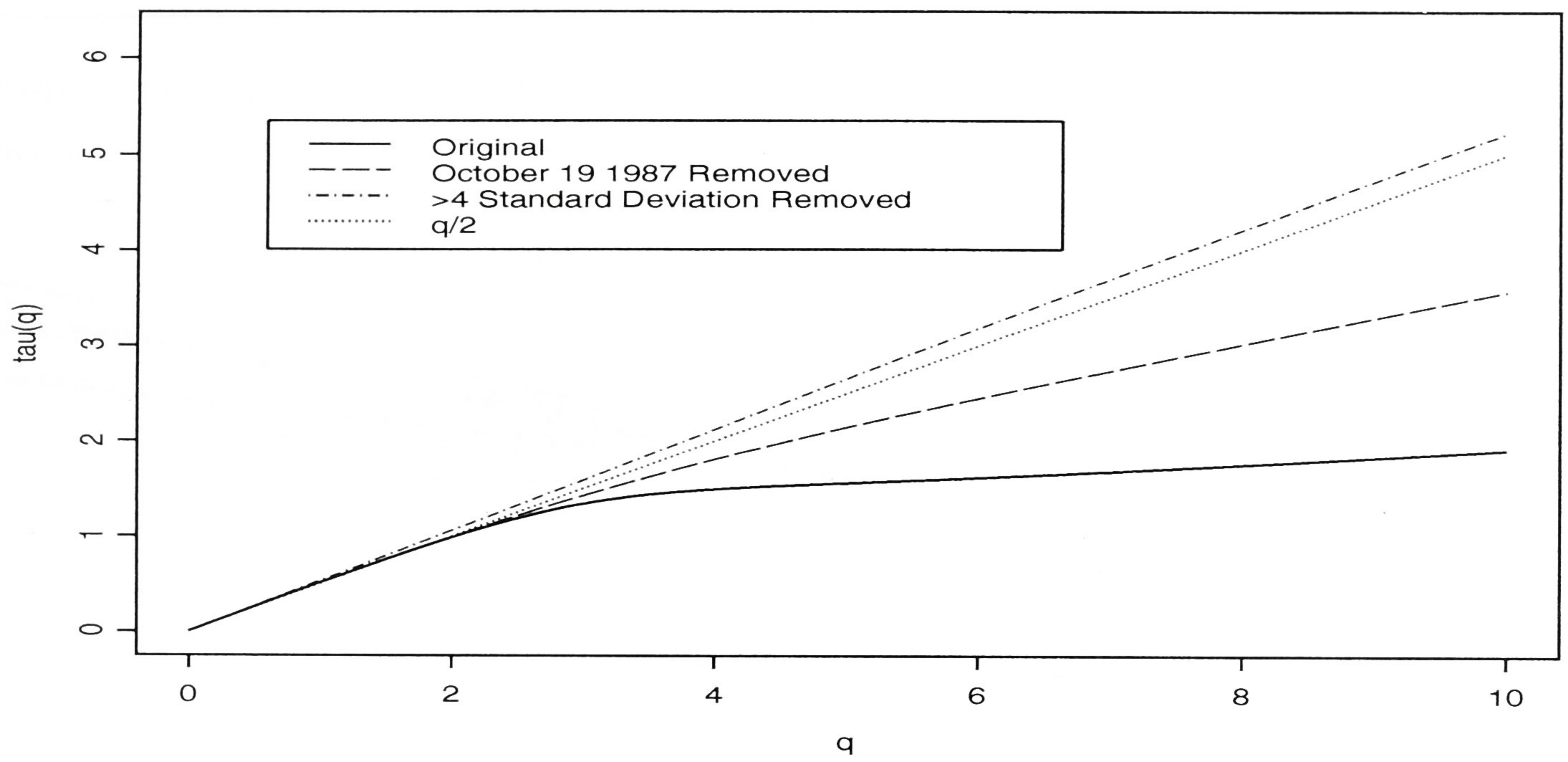

Figure 3.4: Scaling Function of S & P 500 Returns

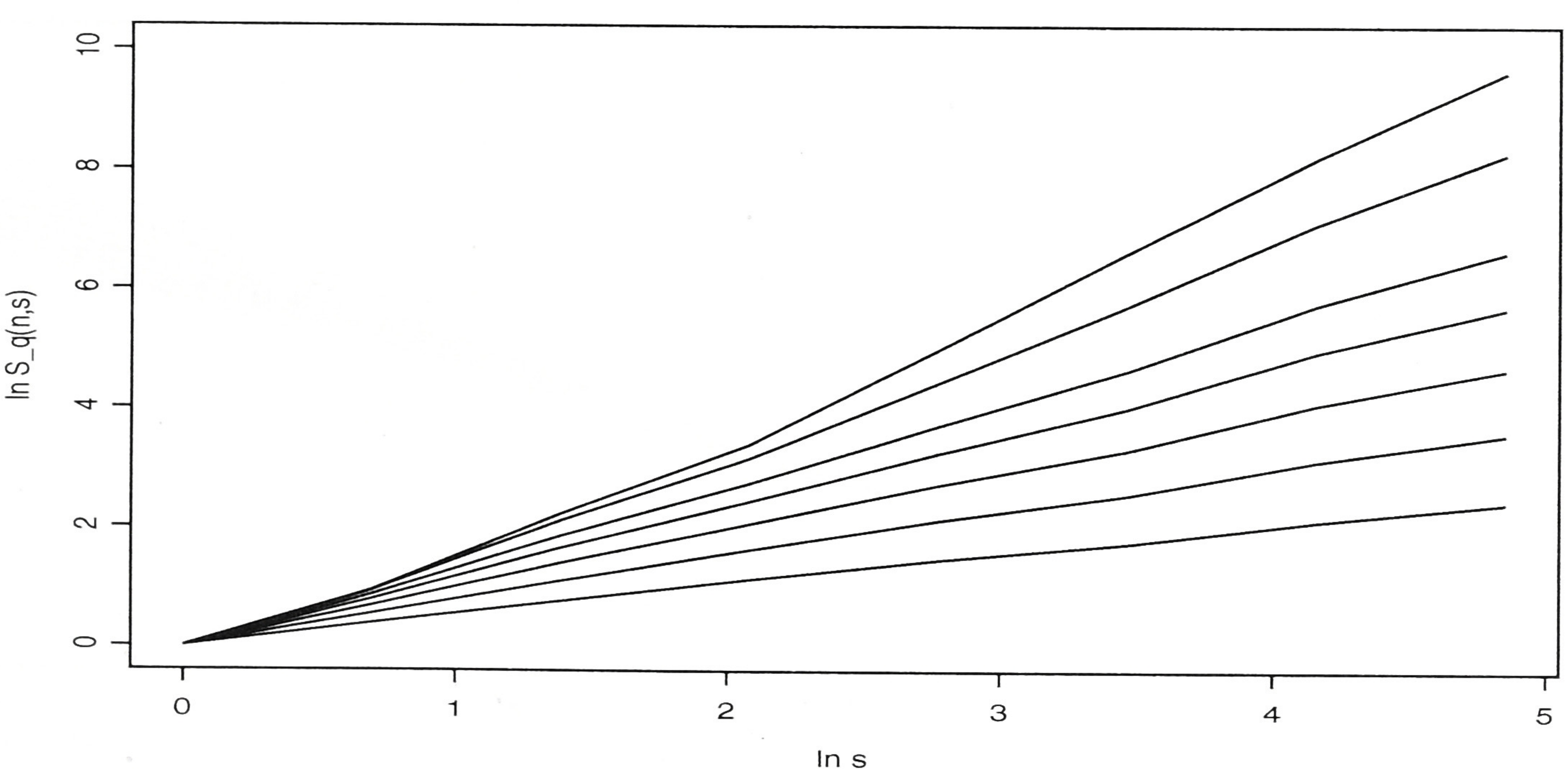

Figure 3.5: Partition Function with $q = 1.0, 1.5, 2.0, 2.5, 3.0, 4.0, 5.0$.

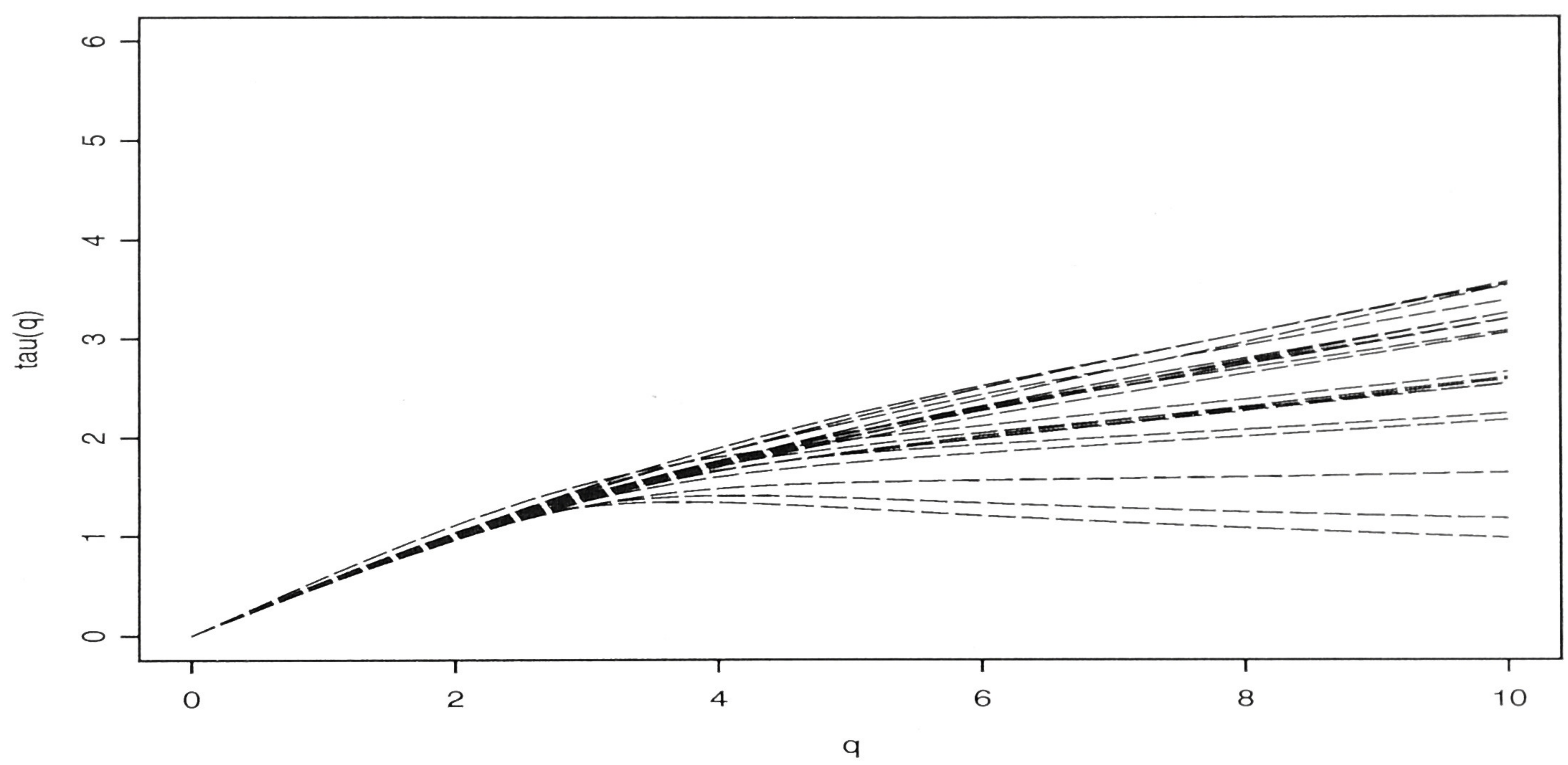

Figure 3.6: Scaling Function for t4 Simulations

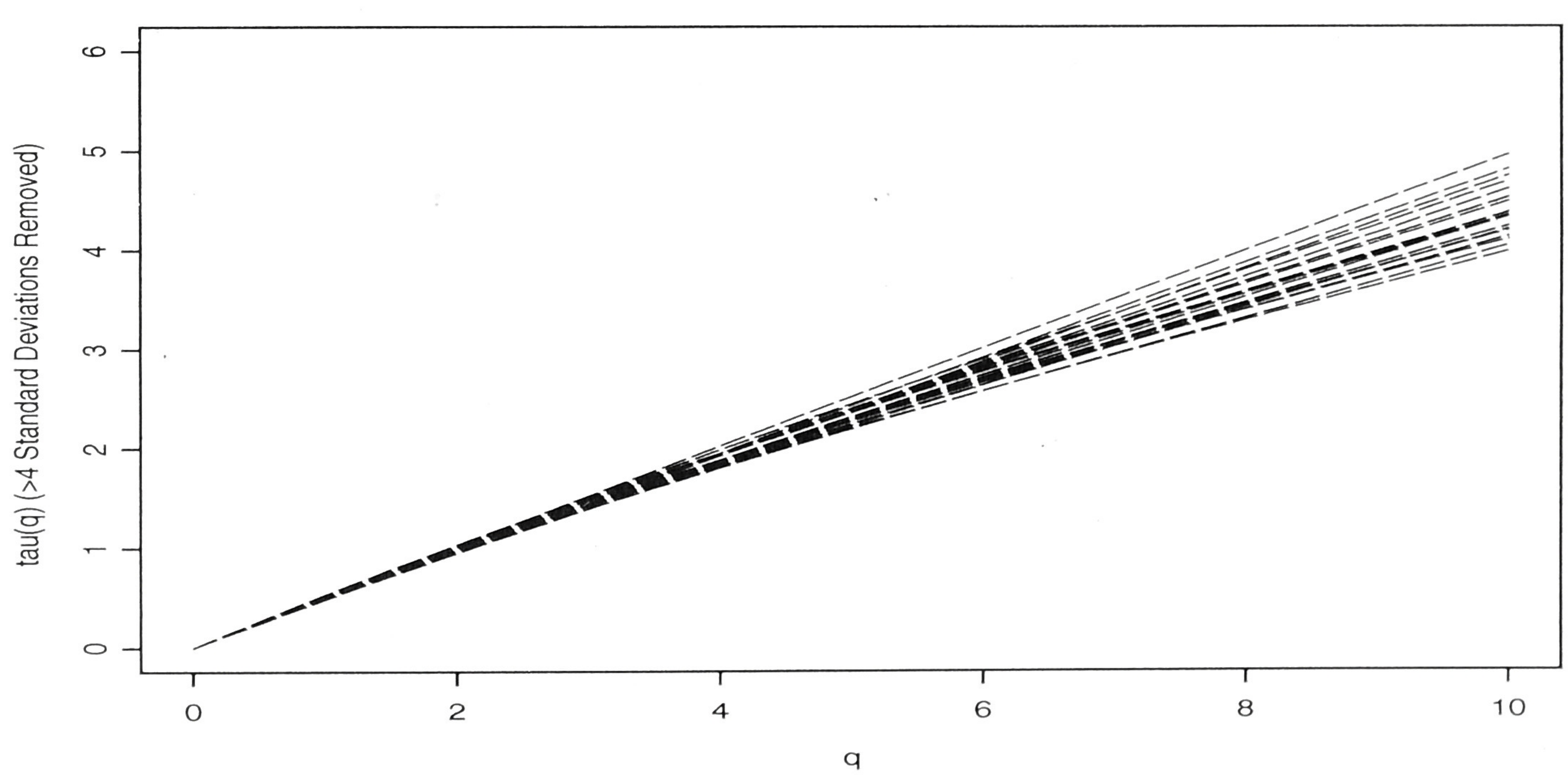

Figure 3.7: Scaling Function for t4 Simulations (Extreme values removed)

3.3.1 Theorem. Let X be a random variable with $EX = 0$ such that the distribution function of $|X|$ has regularly varying tail of order $-\alpha$ where $\alpha > 2$, that is

$$P(|X| > x) = x^{-\alpha}L(x) \tag{3.18}$$

where $L(x)$ is slowly varying. Then for an iid sequence with distribution X and $q > \alpha$, for each $s \in (0,1)$,

$$\frac{\ln S_q(n,n^s)}{\ln n} \xrightarrow{p} \max\{s + \frac{q}{\alpha} - 1, \frac{sq}{2}\} \tag{3.19}$$

as $n \to \infty$.

Proof. Let $\epsilon > 0$, $Y_i = X_i I(|X_i| > n^{\frac{1}{\alpha}+\epsilon})$ and $Z_i = X_i - Y_i + EY_i$. Now

$$\begin{aligned}
E|Y| &\leq \int_0^\infty P(|Y| > x)dx \\
&= \int_{n^{\frac{1}{\alpha}+\epsilon}}^\infty P(|X| > x)dx + n^{\frac{1}{\alpha}+\epsilon}P(|X| > n^{\frac{1}{\alpha}+\epsilon}) \\
&= \int_{n^{\frac{1}{\alpha}+\epsilon}}^\infty x^{-\alpha}L(x)dx + n^{\frac{1}{\alpha}+\epsilon}L(n^{\frac{1}{\alpha}+\epsilon})n^{-\alpha(\frac{1}{\alpha}+\epsilon)} \\
&\leq C_1 n^{\frac{1}{\alpha}-1}
\end{aligned}$$

by Karamata's Lemma (see [Embrechts et al., 1997]). Also $EZ^2 = E(X-Y)^2 - (E(X-Y))^2 \leq EX^2$. Again using Karamata's Lemma,

$$\begin{aligned}
E|Z|^q &\leq 2^{q-1}E|X-Y|^q + 2^{q-1}(E|Y|)^q \\
&= 2^{q-1}\int_0^{n^{q(\frac{1}{\alpha}+\epsilon)}} P(|X-Y|^q > x)dx + 2^{q-1}(E|Y|)^q \\
&\leq 2^{q-1}\int_0^{n^{q(\frac{1}{\alpha}+\epsilon)}} L(x^{\frac{1}{q}})x^{-\frac{\alpha}{q}}dx + 2^{q-1}(E|Y|)^q \\
&\leq C_2 n^{\frac{q}{\alpha}-1+\epsilon q}.
\end{aligned}$$

Now using Rosenthal's inequality,

$$\begin{aligned}
E|\sum_{i=1}^{\lfloor n^s \rfloor} Z_i|^q &\leq C(q)\left(\sum_{i=1}^{\lfloor n^s \rfloor} E|Z_i|^q + (\sum_{i=1}^{\lfloor n^s \rfloor} EZ_i^2)^{q/2}\right) \\
&\leq C(q)\left(n^s C_2 n^{\frac{q}{\alpha}-1+\epsilon q} + n^{\frac{sq}{2}}E(X^2)^{q/2}\right) \\
&\leq C_3 n^{\max\{s+\frac{q}{\alpha}-1+\epsilon q, \frac{sq}{2}\}}.
\end{aligned}$$

Finally,

$$\begin{aligned}
E|\sum_{i=1}^{\lfloor n^s \rfloor} X_i - Y_i|^q &\leq 2^{q-1}E|\sum_{i=1}^{\lfloor n^s \rfloor} Z_i|^q + 2^{q-1}E|n^s EY|^q \\
&\leq 2^{q-1}C_3 n^{\max\{s+\frac{q}{\alpha}-1+\epsilon q, \frac{sq}{2}\}} + 2^{q-1}C_1^q n^{q(s+\frac{1}{\alpha}-1)} \\
&\leq C_4 n^{\max\{s+\frac{q}{\alpha}-1+\epsilon q, \frac{sq}{2}\}}.
\end{aligned}$$

Then for $\delta > 0$,

$$\begin{aligned}
& P\left(\frac{\ln S_q(n,n^s)}{\ln n} > \max\{s+\frac{q}{\alpha}-1, \frac{sq}{2}\} + \epsilon q + \delta\right) \\
\leq\ & P\left(n^{s-1}\sum_{i=1}^{\lfloor n^{1-s}\rfloor}\left|\sum_{j=1}^{\lfloor n^s\rfloor} X_{n^s(i-1)+j} - Y_{n^s(i-1)+j}\right|^q > n^{\max\{s+\frac{q}{\alpha}-1,\frac{sq}{2}\}+\epsilon q+\delta}\right) \\
& +P(\max_{1\leq i\leq n}|X_i| < n^{\frac{1}{\alpha}+\epsilon}) \\
\leq\ & \frac{E|\sum_{i=1}^{n^s} X_i - Y_i|^q}{n^{\max\{s+\frac{q}{\alpha}-1,\frac{sq}{2}\}+\epsilon q+\delta}} + n^{-\alpha\epsilon}L(n^{\frac{1}{\alpha}+\epsilon}) \\
\leq\ & \frac{C_4 n^{\max\{s+\frac{q}{\alpha}-1+\epsilon q, \frac{sq}{2}\}}}{n^{\max\{s+\frac{q}{\alpha}-1,\frac{sq}{2}\}+\epsilon q+\delta}} + n^{-\alpha\epsilon}L(n^{\frac{1}{\alpha}+\epsilon}) \to 0
\end{aligned}$$

and so

$$P\left(\frac{\ln S_q(n,n^s)}{\ln n} > \max\{s+\frac{q}{\alpha}-1, \frac{sq}{2}\} + q\epsilon + \delta\right) \to 0 \tag{3.20}$$

as $n \to \infty$, which establishes the upper bound. We will prove the lower bound in two parts. By the Central Limit Theorem for large n

$$P\left(\left|\sum_{i=1}^{\lfloor n^s\rfloor} X_i\right| > n^{s/2}(EX^2)^{1/2}\right) > \frac{1}{4}.$$

Then

$$\begin{aligned}
& P\left(\frac{\ln S_q(n,n^s)}{\ln n} < \frac{sq}{2} - \epsilon\right) \\
=\ & P\left(n^{s-1}\sum_{i=1}^{\lfloor n^{1-s}\rfloor}\left|\sum_{j=1}^{\lfloor n^s\rfloor} X_{n^s(i-1)+j}\right|^q < n^{\frac{sq}{2}-\epsilon}\right) \\
\leq\ & P\left(\sum_{i=1}^{\lfloor n^{1-s}\rfloor} I\left(\left|\sum_{j=1}^{\lfloor n^s\rfloor} X_{n^s(i-1)+j}\right| > n^{s/2}(EX^2)^{1/2}\right) < (EX^2)^{-q/2}n^{1-s-\epsilon q}\right) \\
\leq\ & P\left(B(\lfloor n^{1-s}\rfloor, \frac{1}{4}) < (EX^2)^{-q/2}n^{1-s-\epsilon q}\right) \to 0
\end{aligned}$$

where $B(n^{1-s}, \frac{1}{4})$ is the binomial distribution. Hence

$$P\left(\frac{\ln S_q(n,n^s)}{\ln n} < \frac{sq}{2} - \epsilon\right) \to 0 \tag{3.21}$$

as $n \to \infty$. Now we consider the case that $\frac{1}{\alpha} - \epsilon > s/2$. Define $k \in \{1, \ldots, n\}$ such that $|X_k| = \max_{1\leq i\leq n}|X_i|$. By equation (3.18)

$$P\left(|X_k| < 2n^{\frac{1}{\alpha}-\epsilon}\right) \to 0.$$

For some integer t, $k \in \{tn^s+1, tn^s+2, \ldots, (t+1)n^s\} = \mathcal{K}$. Then by the Central Limit Theorem,

$$P\left(\left|\sum_{i\in\mathcal{K}\setminus\{k\}} X_i\right| > n^{\frac{1}{\alpha}-\epsilon}\right) \to 0 \tag{3.22}$$

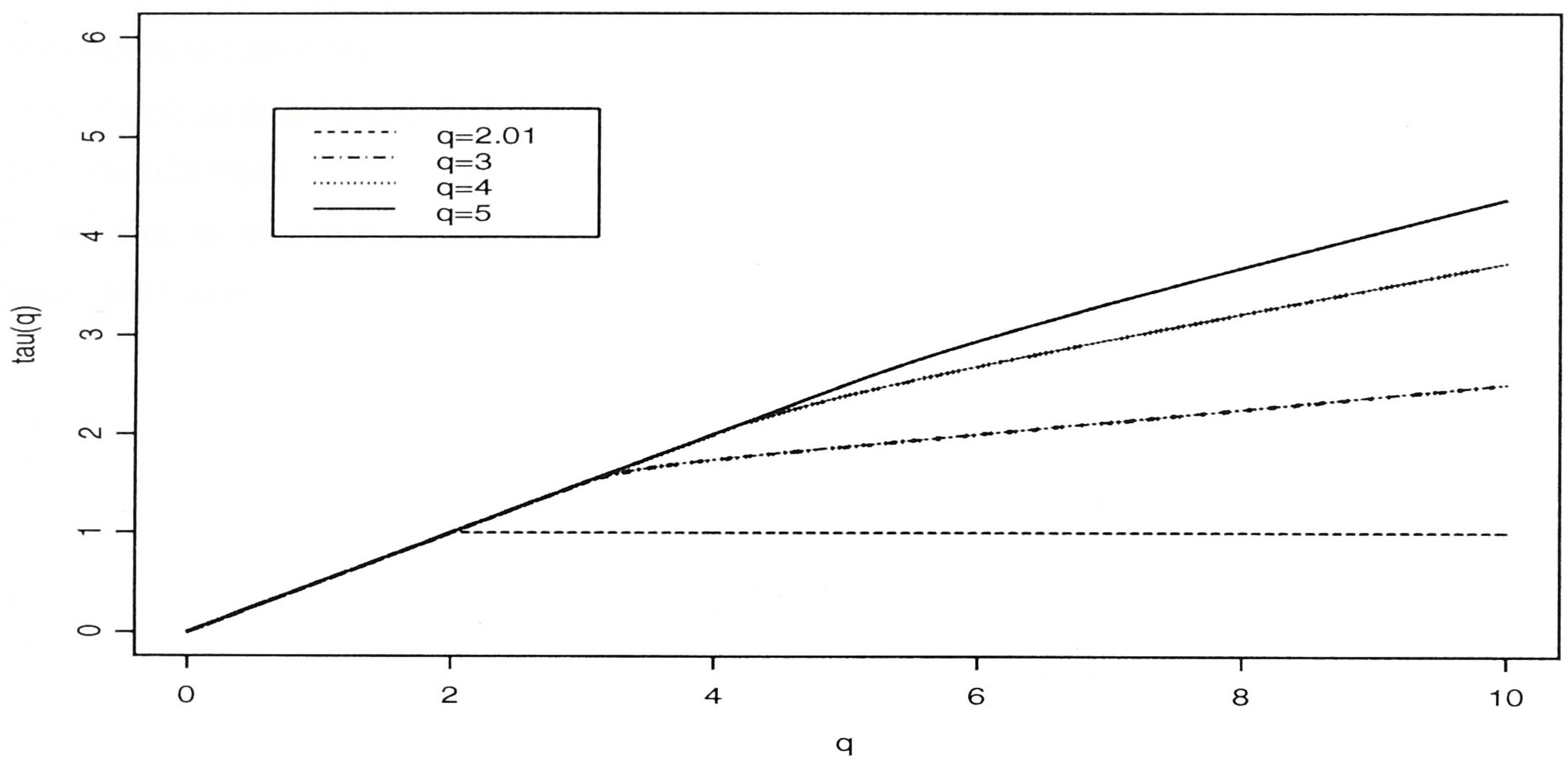

Figure 3.8: Asymptotic Scaling Function

and so

$$P\left(\left|\sum_{i\in\mathcal{K}} X_i\right| > n^{\frac{1}{\alpha}-\epsilon}\right) \to 1$$

as $n \to \infty$. Hence when $s < \frac{2}{\alpha}$

$$P\left(\frac{\ln S_q(n,n^s)}{\ln n} < s - 1 + \frac{q}{\alpha} - q\epsilon\right) \leq P\left(\left|\sum_{i\in\mathcal{K}} X_i\right| < n^{\frac{1}{\alpha}-\epsilon}\right) \to 0. \qquad (3.23)$$

Combining equations (3.20), (3.21) and (3.23) completes the proof. □

When n is large and q is greater than α, $\ln S_q(n,s)$ does not vary linearly with $\ln s$ and as such a linear fit does not make sense. However, if it is naively fitted then

$$\begin{aligned}
\tau(q) &= \frac{1}{(\ln n)^3}\left(12\int_0^{\ln n} s \ln S_q(n,e^s)ds - 6\ln n \int_0^{\ln n} \ln S_q(n,e^s)ds\right) \\
&= 12\int_0^1 s\frac{\ln S_q(n,n^s)}{\ln n}ds - 6\int_0^1 \frac{\ln S_q(n,n^s)}{\ln n}ds.
\end{aligned}$$

As $n \to \infty$, by Theorem 3.3.1,

$$\begin{aligned}
\tau(q) &= 12\int_0^1 s\max\{s + \frac{q}{\alpha} - 1, \frac{sq}{2}\}ds - 6\int_0^1 \max\{s + \frac{q}{\alpha} - 1, \frac{sq}{2}\}ds \\
&= \frac{q}{2} - \frac{2(q-\alpha)^2(3\alpha q - 2\alpha - 4q)}{\alpha^2(q-2)^2}.
\end{aligned}$$

Figure 3.8 shows the limiting estimates for $\alpha = 2.01, 3, 4, 5$.

[Matia et al., 2003] also noted that the distribution of financial returns plays a major role in determining the scaling function. They found that randomly permuting the returns did not have a large effect on the scaling function.

Another explanation for the concavity in the scaling function is given in [Bouchaud et al., 2000]. They develop a process which can be thought of as a subordinator model where the activity time has chi-square increments which are long range dependent. The moments of all orders exist and the process is asymptotically self-similar in the domain of attraction of Brownian motion. Empirical tests find that $\ln S_q(n,t)$ approximately scales linearly with $\ln t$ for a large range of t and the estimated scaling function is concave. This suggests that stochastic volatility as well as heavy tails could produce effects that could be mistaken for multifractality.

3.4 FATGBM

An alternative subordinator model, introduced in [Heyde, 1999] and [Heyde and Liu, 2001], is Fractal Activity Time Geometric Brownian Motion (FATGBM). While BMMT seeks to apply the advanced mathematics of multifractals FATGBM aims to be a minimal descriptive model which incorporates the important features of stock returns.

3.4.1 Definition

In a subordinator model the price process is given by,

$$P_t = \exp(\mu t + \sigma W(T_t))$$

which leaves the activity time T_t to be specified. A discrete approximation for T_t is made by setting

$$T_t - T_{t-1} \stackrel{d}{=} (\frac{1}{\sigma}(X_t - \mu))^2$$

which is derived from the equation

$$d\ln P_t - \frac{dP_t}{P_t} = -\frac{1}{2}\sigma^2 dT_t$$

which follows from Itô's Lemma. By construction the moment index of the activity time is half the moment index of the returns.

The scaling function of the estimated activity time provides strong evidence of non-trivial scaling. Figures 3.9 and 3.10 show the scaling functions for the estimated activity times of the S & P 500 and the Dow components for $0 \leq q \leq 1.2$.

The scaling functions are generally very close to being linear with slopes ranging from $H = 0.7$ to $H = 0.85$. In particular they are consistently above the trivial scaling of slope $\frac{1}{2}$. The scaling function for larger moment, displayed in Figure 3.11, becomes concave. As in the case of the returns this can be explained by the effect of extreme values from the infinite moments. The earlier onset of concavity, as compared with the returns, is explained by the lower moment index.

To incorporate this empirical scaling, the activity time in FATGBM has the asymptotic self-similarity property

$$c^{-H}(T_{ct} - ct) \xrightarrow{d} R(t)$$

in finite dimensional distributions as $c \to \infty$ where $R(t)$ is an H-self-similar process.

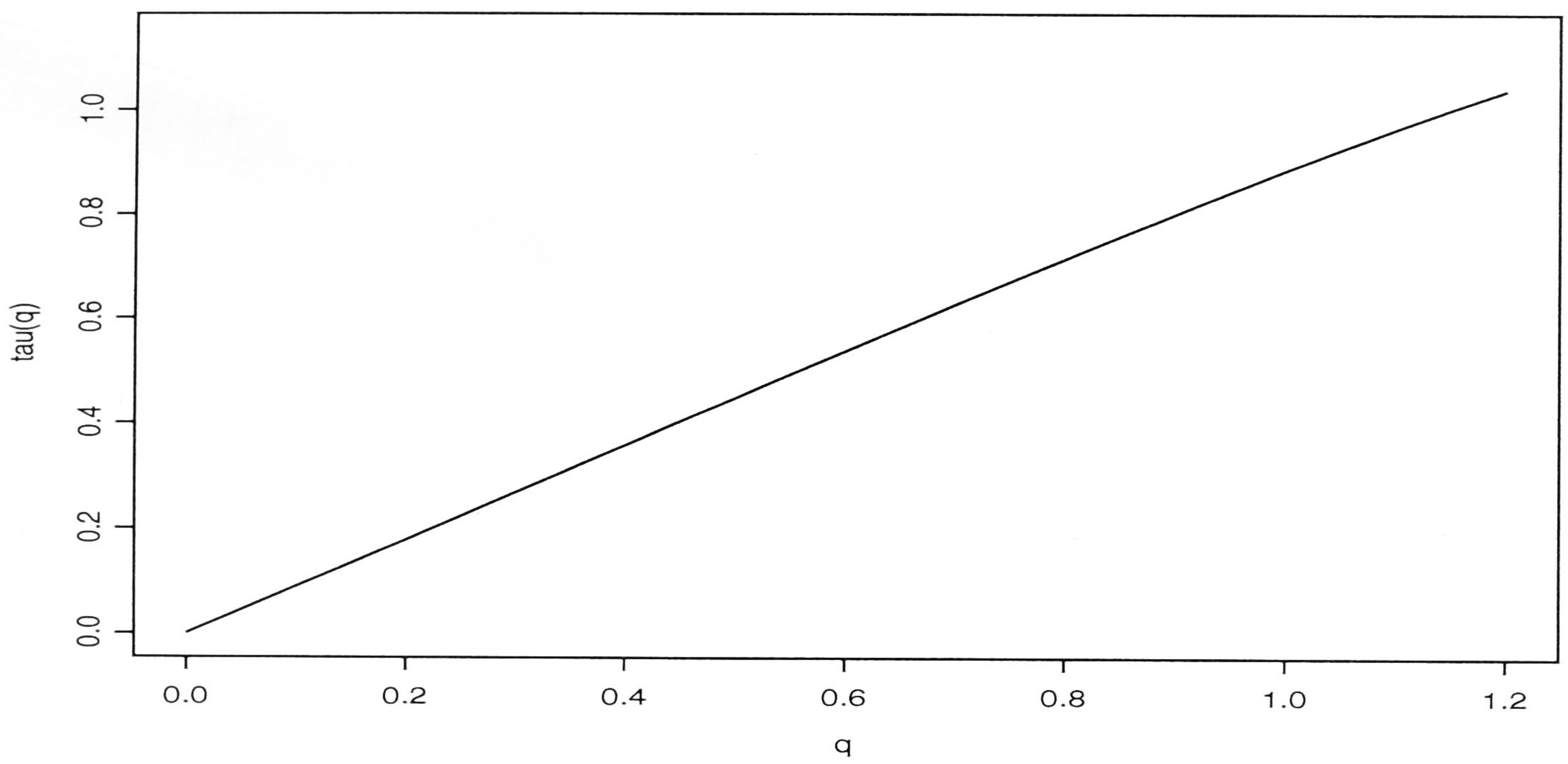

Figure 3.9: Scaling Function of S & P 500 Activity Time

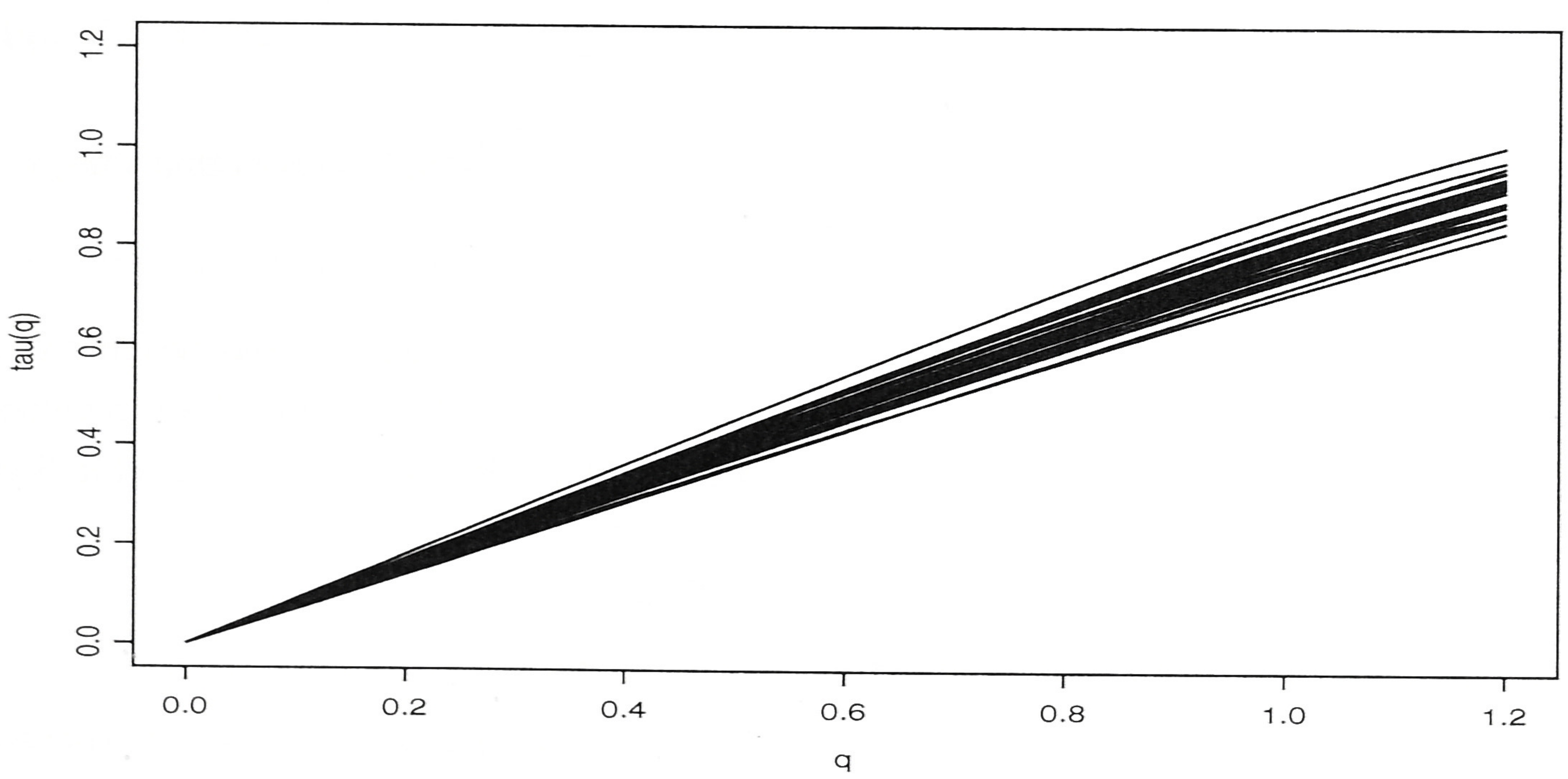

Figure 3.10: Scaling Function of DJIA Components Activity Time

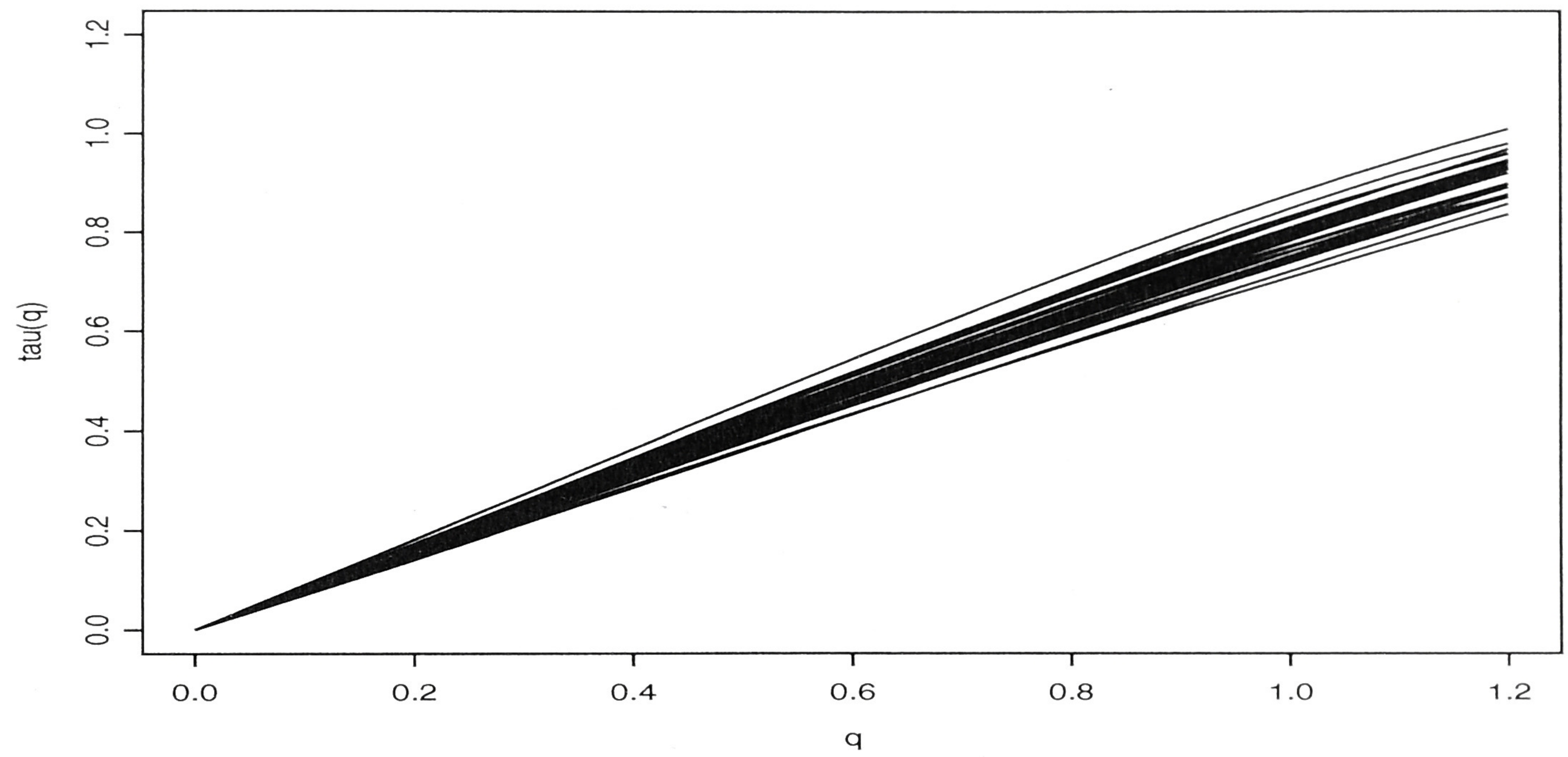

Figure 3.11: Scaling Function of DJIA Components Activity Time

The activity time is also used to incorporate heavy tails into the model. When FATGBM has returns with Student t distributions with ν degrees of freedom the distribution of T_1 is an inverse gamma distribution. [Heyde and Leonenko, 2005] gave a construction for their fractal activity time as follows. For ν an integer and $\frac{1}{2} < H < 1$ let $\eta_1(s), \ldots, \eta_\nu(s)$ be independent stationary Gaussian processes with zero mean and covariance function $p_\eta(t) = (1+t^2)^{-\frac{1-H}{2}}$. Then the activity time is defined as

$$T_n = \sum_{s=1}^{n} \frac{\nu - 2}{\sum_{j=1}^{\nu} \eta_j^2(s)}. \tag{3.24}$$

Then the returns have t_ν distributions. [Heyde and Leonenko, 2005] showed that when $\nu > 4$,

$$c^{-H}(T_{ct} - ct) \xrightarrow{d} CR_\infty(t)$$

in finite dimensional distributions as $c \to \infty$ where R_∞ is the sum of ν independent H-self-similar Rosenblatt processes. The proof can not be extended to $\nu \leq 4$ as the it requires T_n to have finite variance. This problem inspired the work in Chapter 4 and is solved in Section 4.3.

3.4.2 Activity Times for General ν

One drawback of the above definition of fractal activity time is that it has no convenient extension to non-integer values of ν which may be necessary to achieve a good fit to the data. Here we suggest an alternative which has Student t distributed returns and asymptotically self-similar activity time.

Let $R\Gamma(\beta, \alpha)$ denote the inverse gamma distribution with density function $\frac{\alpha^\beta}{\Gamma(\beta)} x^{-\beta-1} e^{-\alpha/x}$ which has moment index β. Let F_ν be the characteristic function of the distribution $[(\frac{\nu}{2}-1)/(\frac{\nu}{2})]R\Gamma(\frac{\nu}{2},\frac{\nu}{2})$ and let Φ be the distribution function of the normal distribution.

To extend this to general ν we take

$$f_\nu(x) = F_\nu^{-1}[\Phi(x)]. \tag{3.25}$$

Let $X(t)$ be a stationary Gaussian process with zero mean and covariance function $p_X(t) = (1+t^2)^{-(1-H)}$ for some $\frac{1}{2} < H < 1$. Then setting

$$T_n = \sum_{s=1}^{n} f_\nu(X(s)) \tag{3.26}$$

as the activity time T_1 has distribution $[(\frac{\nu}{2}-1)/(\frac{\nu}{2})]R\Gamma(\frac{\nu}{2},\frac{\nu}{2})$ so $W(T_1)$ has a Student t distribution with ν degrees of freedom. As f_ν is monotone increasing it follows that $E(f_\nu(X(s))X(s)) > 0$ so by Theorem 1.1.8 when $\nu > 4$

$$c^{-H}(T_{ct} - ct) \xrightarrow{d} CB_H(t)$$

as $c \to \infty$ where $C = E(f_\nu(X(s))X(s))$ and $B_H(t)$ is fractional Brownian motion. It is also asymptotically self-similar when $2 < \nu \leq 4$ which will be proved by Theorem 4.1.1.

3.4.3 Concentration of Extreme Values

While the fractal activity time defined in equation (3.24) is long-range dependent and has strong correlations it does not have large clusters of extreme values. A large extreme value is not likely to be followed by another one of proportional size as is shown in the next result.

3.4.1 Proposition. Let T_n be fractal activity time defined in equation (3.24) and let $\mathcal{F}_s$ be the filtration generated by $(\eta_1(s), \ldots, \eta_\nu(s))$. For some $\sigma > 0$, $\eta_j(s) \stackrel{d}{=} N(E(\eta_j(s)|\mathcal{F}_{s-1}), \sigma^2)$. Then when $\nu > 2$

$$P(T_s - T_{s-1} > x|\mathcal{F}_{s-1}) \leq C_1 x^{-\nu/2} \tag{3.27}$$

where $C_1 = (2(\nu-2))^{\nu/2}(\sigma^2\pi)^{-\nu/2}$.

Proof. We directly estimate,

$$\begin{aligned}
P(T_s - T_{s-1} > x|\mathcal{F}_{s-1}) &= P(\frac{\nu-2}{\sum_{i=1}^{\nu}\eta_i^2(s)} > x|\mathcal{F}_{s-1}) \\
&= P(\sum_{i=1}^{\nu}\eta_i^2(s) < (\nu-2)x^{-1}|\mathcal{F}_{s-1}) \\
&\leq P(\max_{1\leq i\leq\nu}\eta_i^2(s) < (\nu-2)x^{-1}|\mathcal{F}_{s-1}) \\
&= \prod_{i=1}^{\nu} P(\eta_i(s) < |(\nu-2)|^{1/2}|x|^{-1/2}|\mathcal{F}_{s-1}) \\
&= \prod_{i=1}^{\nu} P(N(E(\eta_i(s)|\mathcal{F}_{s-1}), \sigma^2) < |(\nu-2)|^{1/2}|x|^{-1/2}|\mathcal{F}_{s-1}) \\
&\leq \prod_{i=1}^{\nu} P(N(0,\sigma^2) < |(\nu-2)|^{1/2}|x|^{-1/2}|\mathcal{F}_{s-1}) \\
&= C_1 x^{-\nu/2}
\end{aligned}$$

which proves the result. □

When $\nu > 2$ it follows that

$$\|E(X_s|\mathcal{F}_{s-1})\|_\infty < \infty. \tag{3.28}$$

Even for histories with extreme values the conditional expectation is still bounded. As such the extreme values of this version of FATGBM tend to be spread out. The property of extreme values being spread out is characteristic of equities and foreign exchange returns but extreme values of index data tend to appear in clusters.

3.5 Conclusion

Both BMMT and FATGBM are credible models which have sample paths closely resembling real financial returns. They both exhibit heavy tails and have complex scaling behavior.

Since BMMT is based on multifractals it lacks a satisfactory definition for large t and for some data sets this may be prohibitively small. The evidence for a multifractal model is largely associated with concavity of the scaling function. When q is small $\tau(q) \approx \frac{1}{2}q$, the scaling for independent increments. Concavity appears only for larger q and this can be explained by the effect of extreme values.

By contrast FATGBM models the self-similarity in the activity time which is significantly removed from the independent case. Modelling the activity time with self-similarity appears to be the right approach.

Chapter 4

Noncentral Limit Theorem for Infinite Variance Functionals

4.1 Introduction

A large number of processes exhibit both heavy tailed distributions and some form of long memory. These two properties can interact in many complex ways which no single process or theorem can hope to explain. One such class of processes are moving average processes with heavy tailed innovations such as fractional stable motion. As a direct result of their construction the extreme values of these processes appear in small clusters. Multifractals also have concentrated extreme values. It is, however, possible for a process to exhibit long-memory and heavy tails but to have extreme values which are asymptotically independent. This behavior is characteristic of some stock and foreign exchange returns. In this situation the balance between the long range dependence and the heavy tails determines the long run distribution of the process.

This phenomena can emerge from very simple models. In this case we examine functionals of long range dependent Gaussian sequences which are in the domain of attraction of a non-Gaussian stable law with finite mean. A simple example is the process $Y_i = f(X_i)$ where X_i are the increments of fractional Brownian motion and $f(x) = |x|^{-\frac{1}{\alpha}}$ with $1 < \alpha < 2$. Surprisingly even though the Gaussian sequence has a strong dependence structure the extreme values are asymptotically independent.

In this chapter we prove a new limit theorem for these functionals of long-range dependent Gaussian sequences. This result can be seen as an extension of Theorem 1.1.8, the finite variance case which was solved in [Dobrushin and Major, 1979] and [Taqqu, 1979]. As in Theorem 1.1.8 we get a dichotomy between a central and a noncentral limit theorem depending on how strong the long range dependence is and on the Hermite index. However, in our result the limit also depends on a third parameter, the moment index of the distribution. Depending on the choice of these three parameters the limit is either a long range dependent Hermite process with moments of all orders or a non-Gaussian stable law with infinite variance.

The major difficulty in extending the previous result was the lack of a Gaussian Hilbert space decomposition. Our techniques exploit the asymptotic independence of the extreme values by taking a conditional expectation which has finite variance. These can be handled using Theorem 1.1.8. The remainder is short range dependent and we show that it converges to α-stable Lévy motion. This approach may be applicable in other situations with asymptotically independent extreme values.

Our original interest in this model was to understand the behavior of the fractal activity time from [Heyde and Leonenko, 2005] when it has infinite variance. As noted earlier there is significant evidence that some financial returns do not have fourth finite moments so their activity times must be modelled with infinite variance processes. This underlines the importance of limit theorems for infinite variance long range dependent processes. Our method of proof can be modified to explain the asymptotic behavior of the fractal activity time. We now state the result.

4.1.1 Theorem. Let X_i be a stationary sequence of Gaussian random variables such that

$$EX_1X_i \sim \beta i^{2H-2}. \tag{4.1}$$

Let $f:\mathbb{R}\mapsto\mathbb{R}$ satisfy

$$E(f(X))=0, \quad P(f(X)>x)\sim b_+x^{-\alpha}, \quad P(f(X)<-x)\sim b_-x^{-\alpha}, \quad b_++b_->0 \tag{4.2}$$

as $x\to\infty$ and with $1<\alpha<2$. Then there exists a positive integer d and a function g such that $g(E(X_i|\mathcal{F}_{i-d}))=E(f(X_i)|\mathcal{F}_{i-d})$ and $Eg(E(X_i|\mathcal{F}_{i-d}))^2<\infty$. Let κ be the Hermite rank of g. Then if $1-\kappa(1-H)>\frac{1}{\alpha}$

$$n^{-(1-\kappa(1-H))}\sum_{i=1}^{\lfloor nt\rfloor} f(X_i)\xrightarrow{d} CR_{\kappa,1-\kappa(1-H)} \tag{4.3}$$

where $R_\kappa(t)$ is the κ-th Hermite process. But if $1-\kappa(1-H)<\frac{1}{\alpha}$ then

$$\left(\frac{(b_++b_-)\Gamma(2-\alpha)}{\alpha-1}\cos(\frac{\pi(2-\alpha)}{2})\right)^{-\frac{1}{\alpha}} n^{-\frac{1}{\alpha}}\sum_{i=1}^{\lfloor nt\rfloor} f(X_i)\xrightarrow{d} R^*(t) \tag{4.4}$$

where $R^*(t)$ is α-stable Lévy motion with $R^*(1)\stackrel{d}{=}S_\alpha(1,\frac{b_+-b_-}{b_++b_-},0)$ following the notation of [Samorodnitsky and Taqqu, 1994].

As in Theorem 1.1.8 there is a cutoff between the central and non-central limits theorems, however, it is shifted from $1-\kappa(1-H)=\frac{1}{2}$ to $1-\kappa(1-H)=\frac{1}{\alpha}$. With the heavy tails stronger long-range dependence, i.e. larger H is required to overcome the central limit theorems convergence.

The proof of this result is motivated by two observations. The first is that even with long-range dependence consecutive or clustered extreme values are unlikely. This was observed in the case of fractal activity time in the previous chapter but holds true more generally as can be seen in Lemma 4.2.4. The second observation is that for some d, $E(f(X_i)|\mathcal{F}_{i-d})$ has finite variance and so Theorem 1.1.8 can be applied. The proof follows by splitting $f(X_i)$ into

$f(X_i)I(|X_i| \geq cn^{\frac{1}{\alpha}})$, $E(f(X_i)|\mathcal{F}_{i-d})$ and two remainder terms and proving limit theorems separately.

To extend the concept to infinite variance sequences [Heyde and Yang, 1997] defined long range dependence sequences in the sample Allen variance sense as those sequences X_i such that

$$\frac{\left(\sum_{i=1}^{n} X_i\right)^2}{\sum_{i=1}^{n} X_i^2} \xrightarrow{p} \infty. \tag{4.5}$$

as $n \to \infty$. It is a simple consequence of our results that $f(X_i)$ is long range dependent in the sample Allen variance sense if $1 - \kappa(1-H) > \frac{1}{\alpha}$ and short range dependent when $1 - \kappa(1-H) < \frac{1}{\alpha}$.

Proposition 3 of [Heyde, 2002] showed that when X_i are the increments of fractional Brownian motion with $\frac{3}{4} \leq H < 1$ and $r > 0$ then

$$n^{1-2H} \sum_{i=1}^{n} (|X_i|^r - E|X_i|^r) \xrightarrow{d} CR_{2,2H-1}(1) \tag{4.6}$$

as $n \to \infty$. With Theorem 4.1.1 we can extend this result to show that equation (4.6) holds when $\frac{1}{1-2H} < r < 0$ but that

$$n^{1-2H} \sum_{i=1}^{n} (|X_i|^r - E|X_i|^r) \xrightarrow{d} CR^*(t) \tag{4.7}$$

when $-1 < r < \frac{1}{1-2H}$.

Interestingly [Vaiciulis, 2003] proved a limit theorem for a different sequence but arrived at the same limit. The sequence in question was a polynomial function of long-range dependent moving average processes with heavy tailed innovations.

Section 2 gives the proof of Theorem 4.1.1 while Section 3 discusses applications.

4.2 Proof

A collection of lemmas are required for the proof. The first is a general result which is a consequence of [Billingsley, 1968] Theorem 4.2 so we state it without proof.

4.2.1 Lemma. For each $1 \leq k \leq K$ let $W_{k,n}$ be a sequence of random variables such that for each $m \geq 1$ we can write

$$W_{k,n} = X_{k,n,m} + Y_{k,n,m} + Z_{k,n,m}.$$

Suppose that

$$Z_{k,n,m} \xrightarrow{d} 0$$

as $n \to \infty$ and

$$\limsup_{n\to\infty} E|Y_{k,n,m}| = h_{k,m}$$

and $h_{k,m} \to 0$ as $m \to 0$. Finally assume that

$$(X_{1,n,m}, X_{2,n,m}, \ldots, X_{K,n,m}) \xrightarrow{d} (X^*_{1,m}, X^*_{2,m}, \ldots, X^*_{K,m})$$

jointly as $n \to \infty$ and that

$$(X^*_{1,m}, X^*_{2,m}, \dots, X^*_{K,m}) \xrightarrow{d} (W_1, W_2, \dots, W_K)$$

jointly as $m \to \infty$. Then

$$(W_{1,n}, W_{2,n}, \dots, W_{K,n}) \xrightarrow{d} (W_1, W_2, \dots, W_K)$$

jointly as $n \to \infty$.

4.2.2 Lemma. For a d such that

$$E(X_i|\mathcal{F}_{i-d}) \stackrel{d}{=} N(0, \theta)$$

and $\theta < \alpha - 1$. Then

$$E(E(f(X_i)|\mathcal{F}_{i-d}))^2 < \infty.$$

4.2.3 Lemma. Let κ be the Hermite rank of $E(X_i|\mathcal{F}_{i-d})$ and suppose that $1-\kappa(1-H) > \frac{1}{2}$. Then

$$\beta^{-\kappa/2} n^{-(1-\kappa(1-H))} \sum_{i=1}^{\lfloor nt \rfloor} E(f(X_i)|\mathcal{F}_{i-d}) \xrightarrow{d} R_{\kappa,1-\kappa(1-H)}(t)$$

as $n \to \infty$ where convergence is in finite dimensional distributions and $R_{\kappa,1-\kappa(1-H)}$ has distribution

$$R_{k,H'}(t) \stackrel{d}{=} C^k \int_{\mathbb{R}^k} \int_0^t \prod_{i=1}^k |s - x_i|^{H_0 - \frac{3}{2}} ds dB(x_1) \dots dB(x_k)$$

where $H_0 = 1 - \frac{1-H'}{k}$ and

$$C = \left(2\Gamma\left(H_0 - \frac{1}{2}\right) \cos\left(\frac{\pi}{2}\left(H_0 - \frac{1}{2}\right)\right)\right)^{-1}.$$

4.2.4 Lemma. Fix a $c > 0$ and $\lambda > 1$. Define

$$Z_{i,l} = \begin{cases} I[c\lambda^{l-1} n^{\frac{1}{\alpha}} \le f(X_i) < c\lambda^l n^{\frac{1}{\alpha}}] & l > 0, \\ I[-c\lambda^{-l} n^{\frac{1}{\alpha}} \le f(X_i) < -c\lambda^{-l-1} n^{\frac{1}{\alpha}}] & l < 0. \end{cases}$$

By equation (4.2) we have $EZ_{i,l} \sim \mu_l n^{-1}$ where $\mu_l = b_l c^{-\alpha} \lambda^{-|l|\alpha}(\lambda^\alpha - 1)$ and $b_l = b_+$ when $l > 0$ and $b_l = b_-$ when $l < 0$. Then for $a < a'$

$$E\Big(\sum_{i=\lfloor an \rfloor + 1}^{\lfloor a'n \rfloor} Z_{i,l} | \mathcal{F}_{\lfloor an \rfloor}\Big) \to \mu_l (a' - a)$$

in probability as $n \to \infty$.

4.2.5 Lemma. For any $c > 0$,

$$n^{-\frac{1}{\alpha}} \sum_{i=1}^{\lfloor nt \rfloor} f(X_i) I(|f(X_i)| \ge c n^{\frac{1}{\alpha}}) \xrightarrow{d} \int_{\mathbb{R} \setminus (-c,c)} \int_0^t y N(dx, dy)$$

in finite dimensional distributions where $N(dx, dy)$ is an independently scattered Poisson measure with control measure

$$n(dx, dy) = \begin{cases} \alpha b_+ |y|^{-\alpha-1} dx dy & y > 0, \\ \alpha b_- |y|^{-\alpha-1} dx dy & y < 0. \end{cases}$$

4.2.6 Lemma. For any $t > 0$ we have

$$n^{-\frac{1}{\alpha}} \sum_{i=1}^{\lfloor nt \rfloor} E(f(X_i) I(|f(X_i)| \geq cn^{\frac{1}{\alpha}}) | \mathcal{F}_{i-d}) - E(f(X_i) I(|f(X_i)| \geq cn^{\frac{1}{\alpha}})) \xrightarrow{d} 0$$

as $n \to \infty$.

4.2.7 Lemma. For any $t > 0$ we have

$$\limsup_{n\to\infty} E(n^{-\frac{1}{\alpha}} \sum_{i=1}^{\lfloor nt \rfloor} f(X_i) I(|f(X_i)| < cn^{\frac{1}{\alpha}}) - E(f(X_i) I(|f(X_i)| < cn^{\frac{1}{\alpha}}) | \mathcal{F}_{i-d}))^2$$
$$\leq d(b_+ + b_-) \frac{\alpha - 1}{2 - \alpha} c^{2-\alpha} t.$$

Now we will give the proofs of the lemmas.

Proof. (Lemma 4.2.2) Let $1 < \beta < \alpha$ such that

$$\theta < \frac{\beta - 1}{3}.$$

Then when $\beta < \beta_1 < \alpha$,

$$\begin{aligned} E|f(X_i)|^{\beta_1} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x)|^{\beta_1} e^{-\frac{x^2}{2}} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |f(x) e^{-\frac{x^2}{2\beta_1}}|^{\beta_1} dx < \infty \end{aligned}$$

and so it follows that $|f(x)e^{-\frac{x^2}{2\beta_1}}|^{\beta_1}$ is integrable and so $h(x) = f(x)e^{-\frac{x^2}{2\beta}}$ is also integrable. For all $x \in \mathbb{R}$,

$$\frac{x^2}{2\beta} - \frac{(x-s)^2}{2(1-\theta)} \leq \frac{s^2}{2(\beta - (1-\theta))}$$

so

$$\begin{aligned} \left| \frac{1}{\sqrt{2(1-\theta)\pi}} \int_{\mathbb{R}} f(x) e^{-\frac{(x-s)^2}{2(1-\theta)}} dx \right| &\leq \frac{1}{\sqrt{2(1-\theta)\pi}} \int_{\mathbb{R}} |h(x)| e^{\frac{x^2}{2\beta}} e^{-\frac{(x-s)^2}{2(1-\theta)}} dx \\ &\leq Ce^{\frac{s^2}{2(\beta-(1-\theta))}}. \end{aligned}$$

Hence

$$\begin{aligned} E(E(f(X_i)|\mathcal{F}_{i-d}))^2 &= \frac{1}{\sqrt{2\theta\pi}} \int_{\mathbb{R}} \left| \frac{1}{\sqrt{2(1-\theta)\pi}} \int_{\mathbb{R}} f(x) e^{-\frac{x^2}{2(1-\theta)}} dx \right|^2 e^{-\frac{s^2}{2\theta}} ds \\ &\leq \frac{1}{\sqrt{2\theta\pi}} \int_{\mathbb{R}} C^2 e^{\frac{2s^2}{2(\beta-(1-\theta))}} e^{-\frac{s^2}{2\theta}} ds \\ &= \frac{C^2}{\sqrt{2\theta\pi}} \int_{\mathbb{R}} e^{\frac{s^2(\theta+1-\beta)}{2\theta(\beta-(1-\theta))}} ds < \infty. \end{aligned}$$

□

Proof. (Lemma 4.2.3) Let $U_i = \theta^{-\frac{1}{2}} E(f(X_i)|\mathcal{F}_{i-d})$ and $W_i = (1-\theta)^{-\frac{1}{2}}(X_i - E(f(X_i)|\mathcal{F}_{i-d}))$ so $EU_i^2 = EW_i^2 = 1$. Define g by

$$g(\theta^{-\frac{1}{2}}x) = \frac{1}{\sqrt{2(1-\theta)\pi}} \int_{\mathbb{R}} f(y) e^{-\frac{(y-x)^2}{2(1-\theta)}} dy.$$

Now $f(X_i) \stackrel{d}{=} f(N(\theta^{\frac{1}{2}} U_i, 1-\theta))$ and so

$$E(f(X_i)|\mathcal{F}_{i-d}) = E(f(N(\theta^{\frac{1}{2}} U_i, 1-\theta))|\mathcal{F}_{i-d}) = g(U_i).$$

Let

$$g(U_i) = \sum_{k=\kappa}^{\infty} g_i h_k(U_i).$$

Let ρ be an even integer such that $1 - \rho(1-H) < \frac{1}{2}$. Then for $k < \rho$, by applying [Holden et al., 1996] Lemma D.1,

$$\begin{aligned} E(\sum_{i=1}^{n} h_k(X_i) - \theta^{\frac{k}{2}} h_k(U_i))^2 &= E(\sum_{i=1}^{n}\sum_{j=0}^{k-1} \binom{k}{j} \theta^{\frac{j}{2}} (1-\theta)^{\frac{k-j}{2}} h_j(U_i) h_{k-j}(W_i))^2 \\ &\leq C_k n \end{aligned}$$

since $EH_j(U_i)H_{k-j}(W_i)H_j(U_l)H_{k-j}(W_l) = 0$ when $|i-l| < d$. Similarly

$$\begin{aligned} E(\sum_{i=1}^{n} h_\rho(U_i))^2 &= E(\sum_{i=1}^{n} h_\rho(X_i) + (h_\rho(U_i) - h_\rho(X_i)))^2 \\ &\leq C_\rho \rho! n. \end{aligned}$$

Then for $k > \rho$,

$$\begin{aligned} E(\sum_{i=1}^{n} h_k(U_i))^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} Eh_k(U_i)h_k(U_j) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} k!(EU_iU_j)^k \\ &\leq \sum_{i=1}^{n}\sum_{j=1}^{n} k!(EU_iU_j)^\rho \\ &= \frac{k!}{\rho!} E(\sum_{i=1}^{n} h_\rho(U_i))^2 \\ &\leq C_\rho k! n. \end{aligned} \tag{4.8}$$

Hence

$$\begin{aligned} E(\sum_{i=1}^{n}\sum_{k=\rho}^{\infty} g_i h_k(U_i))^2 &= \sum_{k=\rho}^{\infty} g_i^2 (\sum_{i=1}^{n} h_k(U_i))^2 \\ &\leq \sum_{k=\rho}^{\infty} g_i^2 C_\rho k! n \\ &= C_\rho n E(\sum_{k=\rho}^{\infty} g_i h_k(U_i))^2. \end{aligned} \tag{4.9}$$

Then

$$E\left(n^{-(1-\kappa(1-H))}\sum_{i=1}^{n}\left[\sum_{k=\kappa}^{\rho-1} g_k(h_k(U_i)-\theta^{-\frac{k}{2}}h_k(X_i))+\sum_{k=\rho}^{\infty} g_i h_k(U_i)\right]\right)^2 \to 0$$

and so by [Taqqu, 1979]

$$\beta^{-\kappa/2}n^{-(1-\kappa(1-H))}\sum_{i=1}^{\lfloor nt\rfloor}\sum_{k=\kappa}^{\rho-1} g_k\theta^{-\frac{k}{2}}h_k(X_i)\xrightarrow{d} g_\kappa\theta^{-\frac{k}{2}}R_{\kappa,1-\kappa(1-H)}(t)$$

which establishes the result. □

Proof. (Lemma 4.2.4) Define

$$g_{n,l}=\frac{1}{\sqrt{2(1-\theta)\pi}}\int_{\mathbb{R}} I[c\lambda^{l-1}n^{\frac{1}{\alpha}}\le f(y)<c\lambda^{l}n^{\frac{1}{\alpha}}]e^{-\frac{(y-x)^2}{2(1-\theta)}}dy,$$

let $\tau_n=(2\ln(n))^{\frac{1}{2}}$ and let $\max((\theta)^{\frac{1}{2}},(1-\theta)^{\frac{1}{2}})<\gamma_1<\gamma 2<1$. Let $\omega_{n,l}$ be the measure of the set $\{y: c\lambda^{l-1}n^{\frac{1}{\alpha}}\le f(y)<c\lambda^{l}n^{\frac{1}{\alpha}}, |y|\le\gamma_2\tau_n\}$. Clearly $0\le g<1$. Then

$$\begin{aligned}\omega_{n,l} &= e^{\frac{(\gamma_2\tau_n)^2}{2}}\int_{-\gamma_2\tau_n}^{\gamma_2\tau_n} I[c\lambda^{l-1}n^{\frac{1}{\alpha}}\le f(y)<c\lambda^{l}n^{\frac{1}{\alpha}}]e^{\frac{-(\gamma_2\tau_n)^2}{2}}dy\\ &\le e^{\frac{(\gamma_2\tau_n)^2}{2}}\int_{\mathbb{R}} I[c\lambda^{l-1}n^{\frac{1}{\alpha}}\le f(y)<c\lambda^{l}n^{\frac{1}{\alpha}}]e^{\frac{-y^2}{2}}dy\\ &= (2\pi)^{\frac{1}{2}}e^{\frac{(\gamma_2\tau_n)^2}{2}}EZ_{i,l}.\end{aligned}$$

Since $e^{\frac{(\gamma_2\tau_n)^2}{2}}=n^{\gamma_2^2}$ and $nEZ_{i,l}\to\mu_l$, $\omega_{n,l}\le c_1 n^{-1+\gamma_2^2}$. Then when $|x|<\gamma_1\tau_n$,

$$\begin{aligned}g_{n,l}(x) &= \frac{1}{\sqrt{2(1-\theta)\pi}}\int_{\mathbb{R}} I[c\lambda^{l-1}n^{\frac{1}{\alpha}}\le f(y)<c\lambda^{l}n^{\frac{1}{\alpha}}]e^{-\frac{(y-x)^2}{2(1-\theta)}}dy\\ &\le \frac{\omega_{n,l}}{\sqrt{2(1-\theta)\pi}}+\frac{2}{\sqrt{2(1-\theta)\pi}}\int_{\gamma_2\tau_n}^{\infty} I[c\lambda^{l-1}n^{\frac{1}{\alpha}}\le f(y)<c\lambda^{l}n^{\frac{1}{\alpha}},\\ &\quad |y|>\gamma_2\tau_n]e^{-\frac{(y-x)^2}{2(1-\theta)}}dy\\ &\le \frac{c_1n^{-1+\gamma_2^2}+4n^{-\frac{(\gamma_2-\gamma_1)^2}{1-\theta}}}{\sqrt{2(1-\theta)\pi}}\\ &= \psi_n\end{aligned}$$

and $\psi_n\to 0$. Then

$$\begin{aligned}E(Z_{i,l}|\mathcal{F}_{i-d})^2 &= \frac{1}{\sqrt{2\theta\pi}}\int_{\mathbb{R}} g_{n,l}(y)^2e^{-\frac{y^2}{2\theta}}dy\\ &\le \psi_n\frac{1}{\sqrt{2\theta\pi}}\int_{-\gamma_1\tau_n}^{\gamma_1\tau_n} g_{n,l}(y)e^{-\frac{y^2}{2\theta}}dy+\frac{2}{\sqrt{2\theta\pi}}\int_{\gamma_1\tau_n}^{\infty}e^{-\frac{y^2}{2\theta}}dy\\ &\le \psi_nEZ_{i,l}+\frac{2}{\sqrt{2\theta\pi}}\int_{\gamma_1\tau_n}^{\infty}e^{-\frac{y^2}{2\theta}}dy.\end{aligned}$$

For large n

$$\frac{2}{\sqrt{2\theta\pi}}\int_{\gamma_1\tau_n}^{\infty}e^{-\frac{y^2}{2\theta}}dy\le e^{-\frac{(\gamma_1\tau_n)^2}{2\theta}}=n^{-\frac{\gamma_1^2}{\theta}}$$

so $\mathrm{var}(E(Z_{i,l}|\mathcal{F}_{i-d})) \leq E(Z_{i,l}|\mathcal{F}_{i-d})^2 \leq c_2 n^{-1-\epsilon}$ where c_2 depends on l but not on n. Let c_3 be a constant such that,

$$\max_{1\leq k\leq \rho-1} \sup_{|x|\leq t} |h_k(x)| \leq c_3 t^{\rho-1}.$$

As $E(Z_{i,l}|\mathcal{F}_{i-d}) = g_{n,l}(E(Z_{i,l}|\mathcal{F}_{i-d})) = g_{n,l}(\theta^{\frac{1}{2}}U_i)$ we have,

$$\begin{aligned}
|Eh_k(U_i)E(Z_{i,l}|\mathcal{F}_{i-d})| &= \left|\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h_k(y) g_{n,l}(\theta^{\frac{1}{2}}y)^2 e^{-\frac{y^2}{2}}\,dy\right| \\
&\leq \frac{c_3(\ln n)^{\rho-1}}{\sqrt{2\pi}}\int_{-\ln n}^{\ln n} g_{n,l}(\theta^{\frac{1}{2}}y)^2 e^{-\frac{y^2}{2}}\,dy \\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{\ln n}^{\infty} h_k(y)e^{-\frac{y^2}{2}}\,dy \\
&\leq \frac{c_3(\ln n)^{\rho-1}EZ_{i,l}}{\sqrt{2\pi}} + \frac{1}{\sqrt{2\pi}}\int_{\ln n}^{\infty} h_k(y)e^{-\frac{y^2}{2}}\,dy.
\end{aligned}$$

For large n, $\frac{1}{\sqrt{2\pi}}\int_{\ln n}^{\infty} h_k(y)e^{-\frac{y^2}{2}} \leq n^{-2}$ so for some c_4

$$|Eh_k(U_i)E(Z_{i,l}|\mathcal{F}_{i-d})| \leq c_4(\ln n)^{\rho-1}n^{-1}$$

when $k < \rho$. Now

$$\begin{aligned}
& E(\sum_{i=\lfloor an\rfloor+d}^{\lfloor a'n\rfloor}\sum_{k=1}^{\rho-1}\frac{1}{k!}E(h_k(U_i)E(Z_{i,l}|\mathcal{F}_{i-d}))h_k(U_i))^2 \\
=\ & \sum_{k=1}^{\rho-1}E(\sum_{i=\lfloor an\rfloor+d}^{\lfloor a'n\rfloor}\frac{1}{k!}E(h_k(U_i)E(Z_{i,l}|\mathcal{F}_{i-d}))h_k(U_i))^2 \\
\leq\ & \sum_{k=1}^{\rho-1}E(\sum_{i=\lfloor an\rfloor+d}^{\lfloor a'n\rfloor}\frac{c_4}{k!}(\ln n)^{\rho-1}n^{-1}h_k(U_i))^2 \\
\leq\ & c_5(\ln n)^{2(\rho-1)}n^{2H-2} \\
\rightarrow\ & 0
\end{aligned}$$

as $n\to\infty$. By equation (4.9)

$$\begin{aligned}
& E(\sum_{i=\lfloor an\rfloor+d}^{\lfloor a'n\rfloor}\sum_{k=\rho}^{\infty}\frac{1}{k!}E(h_k(U_i)E(Z_{i,l}|\mathcal{F}_{i-d}))h_k(U_i))^2 \\
\leq\ & nC_\rho\sum_{k=\rho}^{\infty}E(\frac{1}{k!}E(h_k(U_i)E(Z_{i,l}|\mathcal{F}_{i-d}))h_k(U_i))^2 \\
\leq\ & nC_\rho E(Z_{i,l}|\mathcal{F}_{i-d})^2 \\
\leq\ & C_\rho c_2 n^{-\epsilon} \\
\rightarrow\ & 0.
\end{aligned}$$

Hence

$$\sum_{i=\lfloor an\rfloor+d}^{\lfloor a'n\rfloor} E(Z_{i,l}|\mathcal{F}_{i-d}) - E(\sum_{i=\lfloor an\rfloor+d}^{\lfloor a'n\rfloor} Z_{i,l}) \to 0$$

in probability as $n \to \infty$. Clearly

$$\sum_{i=\lfloor an \rfloor+1}^{\lfloor an \rfloor+d-1} Z_{i,l} \to 0$$

in probability as $n \to \infty$ so by the Dominated Convergence Theorem,

$$E(\sum_{i=\lfloor an \rfloor+1}^{\lfloor a'n \rfloor} Z_{i,l}|\mathcal{F}_{\lfloor an \rfloor}) \to \mu_l(a'-a).$$

□

Proof. (Lemma 4.2.5) Fix an M and for each natural number m define

$$V_{m,l} = I\left[\sum_{i=\lfloor \frac{n(m-1)}{M} \rfloor+1}^{\lfloor \frac{nm}{M} \rfloor} Z_{i,l} > 0\right].$$

We will show that

$$P(V_{m,l} = 0|\mathcal{F}_{\lfloor \frac{n(m-1)}{M} \rfloor}) \to \exp(-\mu_l M^{-1}). \tag{4.10}$$

as $n \to \infty$ and that the $V_{m,l}$ are asymptotically independent. Partition $[\frac{m-1}{M}, \frac{m}{M}]$ as $\frac{m-1}{M} = a_0 \leq a_1 \leq \ldots \leq a_k = \frac{m}{M}$. By Lemma 4.2.4

$$\begin{aligned} P(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} = 0|\mathcal{F}_{\lfloor a_j n \rfloor}) &\geq 1 - E(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l}|\mathcal{F}_{\lfloor a_j n \rfloor}) \\ &\to 1 - \mu_l(a_{j+1} - a_j) \end{aligned}$$

in probability as $n \to \infty$. Let $S_k = \sum_{i=\lfloor a_j n \rfloor+1}^{k} Z_{i,l}$. Then by Lemma 4.2.4 again

$$\begin{aligned} P(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} > 0|\mathcal{F}_{\lfloor a_j n \rfloor}) &= \sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} P(Z_{i,l} = 1, S_{i-1} = 0|\mathcal{F}_{\lfloor a_j n \rfloor}) \\ &= E(E(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} I(S_{i-1} = 0)|\mathcal{F}_{\lfloor a_j n \rfloor})) \\ &\geq E(E(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} I(S_{\lfloor a_{j+1} n \rfloor} = 0)|\mathcal{F}_{\lfloor a_j n \rfloor})) \\ &\to P(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} = 0|\mathcal{F}_{\lfloor a_j n \rfloor})\mu_l(a_{j+1} - a_j). \end{aligned}$$

It follows that for all $\epsilon > 0$,

$$P\left(P(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} = 0|\mathcal{F}_{\lfloor a_j n \rfloor}) \geq \mu_l(a_{j+1} - a_j) - (\mu_l(a_{j+1} - a_j))^2 - \epsilon\right) \to 1$$

and

$$P\left(P(\sum_{i=\lfloor a_j n \rfloor+1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} = 0|\mathcal{F}_{\lfloor a_j n \rfloor}) \leq \mu_l(a_{j+1} - a_j) + \epsilon\right) \to 1$$

as $n \to \infty$. Since

$$P(V_{m,l} = 0 | \mathcal{F}_{\lfloor \frac{n(m-1)}{M} \rfloor}) = \prod_{j=0}^{k-1} P\Big(\sum_{i=\lfloor a_j n \rfloor + 1}^{\lfloor a_{j+1} n \rfloor} Z_{i,l} = 0 \Big| \sum_{i=\lfloor a_0 n \rfloor + 1}^{\lfloor a_j n \rfloor} Z_{i,l} = 0, \mathcal{F}_{\lfloor \frac{n(m-1)}{M} \rfloor} \Big)$$

we get equation (4.10) by taking progressively finer partitions of $[\frac{m-1}{M}, \frac{m}{M}]$. This shows that $V_{m,l}$ converges in distribution to a Binomial distribution $B(1, 1-\exp(-\mu_l M^{-1}))$. To show asymptotic independence we need to show that for any finite collection of variables $V_{m_k,l_k}, 1 \leq k \leq K$,

$$P(V_{m_k,l_k} = 0, 1 \leq k \leq K) \to \prod_{k=1}^{K} \exp(-\mu_{l_k} M^{-1}). \tag{4.11}$$

as $n \to \infty$. Suppose first that $m = m_1 = m_2 = \ldots = m_k$ and let

$$V_m^* = I \left[\sum_{i=\lfloor \frac{n(m-1)}{M} \rfloor + 1}^{\lfloor \frac{nm}{M} \rfloor} \sum_{k=1}^{K} Z_{i,l_k} > 0 \right].$$

By adapting the proof of equation (4.10) we get that

$$P(V_m^* = 0 | \mathcal{F}_{\lfloor \frac{n(m-1)}{M} \rfloor}) \to \exp(-\sum_{k=1}^{K} \mu_{l_k} M^{-1}).$$

as $n \to \infty$ and so

$$P(V_{m,l_{m_k}} = 0, 1 \leq k \leq K | \mathcal{F}_{\lfloor \frac{n(m-1)}{M} \rfloor}) \to \prod_{k=1}^{K} \exp(-\mu_{l_k} M^{-1}).$$

as $n \to \infty$. Since the conditional probability converges to a constant in probability equation (4.11) and asymptotic independence follows immediately. Now let $0 = t_0 \leq t_1 \leq \ldots \leq t_K$ be a sequence of times. Let $\widetilde{V}_{k,l} = \sum_{i=\lfloor t_{k-1} n \rfloor + 1}^{\lfloor t_k n \rfloor} Z_{i,l} \geq \sum_{i=\lfloor t_{k-1} M \rfloor + 1}^{\lfloor t_k M \rfloor} V_{m,l}$. Then

$$\sum_{i=\lfloor t_{k-1} M \rfloor + 1}^{\lfloor t_k M \rfloor} V_{m,l} \xrightarrow{d} B(\lfloor t_k M \rfloor - \lfloor t_{k-1} M \rfloor, 1 - \exp(-\mu_l M^{-1}))$$

as $n \to \infty$ and

$$B(\lfloor t_k M \rfloor - \lfloor t_{k-1} M \rfloor, 1 - \exp(-\mu_l M^{-1})) \xrightarrow{d} \text{Poisson}((t_k - t_{k-1})\mu_l)$$

as $M \to \infty$. Noting that $V_{m,l}$ is defined in terms of M we have

$$E\widetilde{V}_{k,l} - E \sum_{i=\lfloor t_{k-1} M \rfloor + 1}^{\lfloor t_k M \rfloor} V_{m,l} \to \mu_l (t_k - t_{k-1}) - (\lfloor t_k M \rfloor - \lfloor t_{k-1} M \rfloor)(1 - e^{-\mu_l M^{-1}})$$

as $n \to \infty$ and

$$\mu_l (t_k - t_{k-1}) - M(t_k - t_{k-1})(1 - e^{-\mu_l M^{-1}}) \to 0$$

as $M \to \infty$. Then by Lemma 4.2.1

$$\widetilde{V}_{k,l} \xrightarrow{d} \text{Poisson}((t_k - t_{k-1})\mu_l)$$

as $n \to \infty$ and the limits are asymptotically independent. We will write this result in terms of a Poisson measure. Let $N(dx, dy)$ be an independently scattered Poisson measure with control measure

$$n(dx, dy) = \begin{cases} \alpha b_+ |y|^{-\alpha-1} dx dy & y > 0, \\ \alpha b_- |y|^{-\alpha-1} dx dy & y < 0. \end{cases}$$

Then for $l > 0$

$$\widetilde{V}_{k,l} \xrightarrow{d} \int_{t_{k-1}}^{t_k} \int_{c\lambda^{l-1}}^{c\lambda^l} N(dx, dy)$$

jointly. By equation (4.2)

$$\sup_{n \geq 1} P\left(\max_{1 \leq i \leq t_k n} |f(X_i)| > rcn^{\frac{1}{\alpha}} \right) \to 0$$

as $r \to \infty$ so

$$\sum_{l=1}^{\infty} c\lambda^{l-1} \widetilde{V}_{k,l} \xrightarrow{d} \int_{t_{k-1}}^{t_k} \int_c^{\infty} \sum_{l=1}^{\infty} c\lambda^{l-1} I(c\lambda^{l-1} \leq y < c\lambda^{l-1}) N(dx, dy)$$

jointly as $n \to \infty$ and as $\lambda \to 1$

$$\int_{t_{k-1}}^{t_k} \int_c^{\infty} \sum_{l=1}^{\infty} c\lambda^{l-1} I(c\lambda^{l-1} \leq y < c\lambda^{l-1}) N(dx, dy) \xrightarrow{d} \int_{t_{k-1}}^{t_k} \int_c^{\infty} c\lambda^{l-1} y N(dx, dy).$$

Then since

$$\sum_{l=1}^{\infty} c\lambda^{l-1} \widetilde{V}_{k,l} \leq n^{-\frac{1}{\alpha}} \sum_{i=\lfloor t_{k-1} n \rfloor + 1}^{\lfloor t_k n \rfloor} f(X_i) I(f(X_i) \geq cn^{\frac{1}{\alpha}}) \leq \lambda \sum_{l=1}^{\infty} c\lambda^{l-1} \widetilde{V}_{k,l}$$

it follows that

$$n^{-\frac{1}{\alpha}} \sum_{i=\lfloor t_{k-1} n \rfloor + 1}^{\lfloor t_k n \rfloor} f(X_i) I(f(X_i) \geq cn^{\frac{1}{\alpha}}) \xrightarrow{d} \int_{t_{k-1}}^{t_k} \int_c^{\infty} y N(dx, dy)$$

as $n \to \infty$ and similarly

$$n^{-\frac{1}{\alpha}} \sum_{i=\lfloor t_{k-1} n \rfloor + 1}^{\lfloor t_k n \rfloor} f(X_i) I(-f(X_i) \geq cn^{\frac{1}{\alpha}}) \xrightarrow{d} \int_{t_{k-1}}^{t_k} \int_{-\infty}^{-c} y N(dx, dy)$$

and the result follows. □

Proof. (Lemma 4.2.6) Let $\max\{1, H\alpha\} < \beta < \alpha$. Since $h_k(U_i)$ has moments of all order, $E|f(X_i)|^{\beta} |h_k(U_i)| < \infty$ and so

$$\begin{aligned}
& |E(E(f(X_i) I(|f(X_i)| \geq cn^{\frac{1}{\alpha}}) | \mathcal{F}_{i-d}) h_k(U_i))| \\
= \; & |E(E(f(X_i) I(|f(X_i)| \geq cn^{\frac{1}{\alpha}}) h_k(U_i) | \mathcal{F}_{i-d}))| \\
= \; & |E(f(X_i) I(|f(X_i)| \geq cn^{\frac{1}{\alpha}}) h_k(U_i))| \\
\leq \; & E((cn^{\frac{1}{\alpha}})^{1-\beta} |f(X_i)|^{\beta} I(|f(X_i)| \geq cn^{\frac{1}{\alpha}}) |h_k(U_i)|) \\
\leq \; & c^{1-\beta} E(|f(X_i)|^{\beta} |h_k(U_i)|) n^{\frac{1}{\alpha} - \frac{\beta}{\alpha}}.
\end{aligned}$$

It follows that

$$E(n^{-\frac{1}{\alpha}}\sum_{i=1}^{nt}\sum_{k=1}^{\rho-1}\frac{1}{k!}E(E(f(X_i)I(|f(X_i)|\geq cn^{\frac{1}{\alpha}})|\mathcal{F}_{i-d})h_k(U_i))h_k(U_i))^2 \leq c_6 n^{2H-\frac{2\beta}{\alpha}}t^{2H}$$

which converges to 0 as $n\to\infty$. By a modification of Lemma 4.2.2 $E(E(|f(X_i)||\mathcal{F}_{i-d}))^2<\infty$ and so by equation (4.9),

$$\begin{aligned}
& E(n^{-\frac{1}{\alpha}}\sum_{i=1}^{nt}\sum_{k=\rho}^{\infty}\frac{1}{k!}E(E(f(X_i)I(|f(X_i)|\geq cn^{\frac{1}{\alpha}})|\mathcal{F}_{i-d})h_k(U_i))h_k(U_i))^2 \\
\leq\ & C_\rho n^{1-\frac{2}{\alpha}}E(E(f(X_i)I(|f(X_i)|\geq cn^{\frac{1}{\alpha}})|\mathcal{F}_{i-d}))^2 \\
\leq\ & C_\rho n^{1-\frac{2}{\alpha}}E(E(|f(X_i)||\mathcal{F}_{i-d}))^2
\end{aligned}$$

which converges to 0 as $n\to\infty$ and the result follows. □

Proof. (Lemma 4.2.7) By Karamata's Lemma (see [Embrechts et al., 1997]),

$$\begin{aligned}
& E(f(X_i)I(|f(X_i)|<cn^{\frac{1}{\alpha}}))^2 \\
=\ & \int_0^{(cn^{\frac{1}{\alpha}})^2} P(f(X_i)^2 I(|f(X_i)|<cn^{\frac{1}{\alpha}})>x)dx \\
=\ & \int_0^{(cn^{\frac{1}{\alpha}})^2} P(|f(X_i)|>x^{\frac{1}{2}})dx-(cn^{\frac{1}{\alpha}})^2P(|f(X_i)|>cn^{\frac{1}{\alpha}}) \\
\sim\ & (b_+ + b_-)(1-\frac{\alpha}{2})^{-1}((cn^{\frac{1}{\alpha}})^2)^{1-\frac{\alpha}{2}}-(b_+ + b_-)(cn^{\frac{1}{\alpha}})^2(cn^{\frac{1}{\alpha}})^{-\alpha} \\
=\ & \frac{\alpha-1}{2-\alpha}(b_+ + b_-)c^{2-\alpha}n^{\frac{2}{\alpha}-1}
\end{aligned}$$

as $n\to\infty$. Clearly

$$E(f(X_i)I(|f(X_i)|<cn^{\frac{1}{\alpha}})-E(f(X_i)I(|f(X_i)|<cn^{\frac{1}{\alpha}})|\mathcal{F}_{i-d}))=0$$

and when $|i-j|\geq d$,

$$\begin{aligned}
& E(f(X_i)I(|f(X_i)|<cn^{\frac{1}{\alpha}})-E(f(X_i)I(|f(X_i)|<cn^{\frac{1}{\alpha}})|\mathcal{F}_{i-d})) \\
& \times(f(X_j)I(|f(X_j)|<cn^{\frac{1}{\alpha}})-E(f(X_j)I(|f(X_j)|<cn^{\frac{1}{\alpha}})|\mathcal{F}_{j-d}))=0
\end{aligned}$$

so the result follows. □

Proof. (Theorem 4.1.1) Observe that $f(X_i)$ can be split up as follows,

$$\begin{aligned}
\sum_{i=1}^{\lfloor nt\rfloor} f(X_i) &= \sum_{i=1}^{\lfloor nt\rfloor} f(X_i|\mathcal{F}_{i-d}) \\
&\quad+\sum_{i=1}^{\lfloor nt\rfloor} f(X_i)I(|f(X_i)|<cn^{\frac{1}{\alpha}})-E(f(X_i)I(|f(X_i)|<cn^{\frac{1}{\alpha}})|\mathcal{F}_{i-d}) \\
&\quad+\sum_{i=1}^{\lfloor nt\rfloor} f(X_i)I(|f(X_i)|\geq cn^{\frac{1}{\alpha}})-Ef(X_i)I(|f(X_i)|\geq cn^{\frac{1}{\alpha}}) \\
&\quad+\sum_{i=1}^{\lfloor nt\rfloor} Ef(X_i)I(|f(X_i)|\geq cn^{\frac{1}{\alpha}})-E(f(X_i)I(|f(X_i)|\geq cn^{\frac{1}{\alpha}})|\mathcal{F}_{i-d}) \\
&= A_1(t)+A_2(t)+A_3(t)+A_4(t).
\end{aligned}$$

If $1-\kappa(1-H) > \frac{1}{\alpha}$ then by Lemmas 4.2.5, 4.2.6 and 4.2.7,

$$n^{-(1-\kappa(1-H))}(A_2(t)+A_3(t)+A_4(t)) \xrightarrow{d} 0$$

as $n \to \infty$ while Lemma 4.2.3 shows that,

$$n^{-(1-\kappa(1-H))}A_1(t) \xrightarrow{d} CR_{\kappa,1-\kappa(1-H)}(t)$$

in finite dimensional distributions as $n \to \infty$ and so

$$n^{-(1-\kappa(1-H))}\sum_{i=1}^{\lfloor nt \rfloor} f(X_i) \xrightarrow{d} CR_{\kappa,1-\kappa(1-H)}(t).$$

On the other hand if $1-\kappa(1-H) < \frac{1}{\alpha}$ then by Lemmas 4.2.3 and 4.2.6,

$$n^{-\frac{1}{\alpha}}(A_1(t)+A_4(t)) \xrightarrow{d} 0$$

as $n \to \infty$. By Lemma 4.2.7,

$$\limsup_{n\to\infty} n^{-\frac{1}{\alpha}}E|A_2(t)-A_2(s)| \leq \sqrt{d\frac{\alpha-1}{2-\alpha}(b_+b_-)c^{2-\alpha}|t-s|}.$$

By Lemma 4.2.5,

$$n^{-\frac{1}{\alpha}}A_3(t) \xrightarrow{d} \int_0^t \int_{\mathbb{R}\backslash(-c,c)} yN(dx,dy) - E\int_0^t \int_{\mathbb{R}\backslash(-c,c)} yN(dx,dy)$$

in finite dimensional distributions as $n \to \infty$ and by Theorem 3.12.2 of [Samorodnitsky and Taqqu, 1994],

$$\left(\frac{(b_+ + b_-)\Gamma(2-\alpha)}{\alpha-1}\cos(\frac{\pi(2-\alpha)}{2})\right)^{-\frac{1}{\alpha}} \int_0^t \int_{\mathbb{R}\backslash(-c,c)} yN(dx,dy) - E\int_0^t \int_{\mathbb{R}\backslash(-c,c)} yN(dx,dy) \xrightarrow{d} R^*(t)$$

in finite dimensional distributions as $c \to 0$ where $R^*(t)$ is alpha stable Lévy motion. Now let $0 = t_0 \leq t_1 \leq \ldots \leq t_K$ be a sequence of times. Set

$$\begin{aligned}
X_{k,n,m} &= n^{-\frac{1}{\alpha}}(A_3(t_k) - A_3(t_{k-1})) \\
Y_{k,n,m} &= n^{-\frac{1}{\alpha}}(A_2(t_k) - A_2(t_{k-1})) \\
Z_{k,n,m} &= n^{-\frac{1}{\alpha}}(A_1(t_k) - A_1(t_{k-1}) + A_4(t_k) - A_4(t_{k-1}))
\end{aligned}$$

with $c = \frac{1}{m}$. Applying Lemma 4.2.1 shows that

$$\left(\frac{(b_+ + b_-)\Gamma(2-\alpha)}{\alpha-1}\cos(\frac{\pi(2-\alpha)}{2})\right)^{-\frac{1}{\alpha}} n^{-\frac{1}{\alpha}}\sum_{i=1}^{\lfloor nt \rfloor} f(X_i) \xrightarrow{d} R^*(t)$$

in finite dimensional distributions as $n \to \infty$ which completes the proof. □

4.3 Applications

Our attempt to understand the convergence of the fractal activity time defined in equation (3.24) when $\nu = 3, 4$ motivated the development of Theorem 4.1.1. However, the theorem can not be directly applied as $T_n - T_{n-1} - 1$ is a function of ν Gaussian random variables. Modifying the proof of the theorem to this example is straightforward and equation (3.28) simplifies many of the estimates. From this modification we get that when $\nu = 3$,

$$(\frac{4}{3})^{-\frac{2}{3}} n^{-\frac{2}{3}} \sum_{i=1}^{\lfloor nt \rfloor} T_i - T_{i-1} - E(T_i - T_{i-1}|\mathcal{F}_{i-1}) \xrightarrow{d} R^*(t) \tag{4.12}$$

where $R^*(t)$ is α-stable Lévy motion with $R^*(1) \stackrel{d}{=} S_\alpha(1,1,0)$. It can also be shown that when $\nu = 4$,

$$n^{-H} \sum_{i=1}^{\lfloor nt \rfloor} T_i - T_{i-1} - E(T_i - T_{i-1}|\mathcal{F}_{i-1}) \xrightarrow{d} 0 \tag{4.13}$$

as $H > \frac{1}{2}$. Applying Theorem 2 of [Arcones, 2000] shows that

$$n^{-H} \sum_{i=1}^{\lfloor nt \rfloor} E(T_i - T_{i-1}|\mathcal{F}_{i-1}) - 1 \xrightarrow{d} CR_\infty(t) \tag{4.14}$$

as $n \to \infty$ where $R_\infty(t)$ is the sum of ν independent Rosenblatt processes. Hence if $H > \frac{2}{\nu}$ then

$$n^{-H}(T_{nt} - nt) \xrightarrow{d} CR_\infty(t)$$

while if $\nu = 3$ and $H < \frac{2}{\nu}$ then

$$(\frac{4}{3})^{-\frac{2}{3}} n^{-\frac{2}{\nu}} (T_{nt} - nt) \xrightarrow{d} R^*(t)$$

in finite dimensional distributions as $n \to \infty$.

On the other hand Theorem 4.1.1 can be directly applied to the modified activity time defined in equation (3.25). If $H > \frac{2}{\nu}$ then

$$n^{-H}(T_{nt} - nt) \xrightarrow{d} CB_H(t)$$

while if $H < \frac{2}{\nu}$ then

$$\left(\frac{(\frac{\nu}{2}-1)^{\frac{\nu}{2}} \Gamma(2-\frac{\nu}{2})}{\frac{\nu}{2}(\frac{\nu}{2}-1)\Gamma(\frac{\nu}{2})} \cos(\frac{\pi(2-\frac{\nu}{2})}{2}) \right)^{-\frac{1}{\alpha}} n^{-\frac{2}{\nu}} (T_{nt} - nt) \xrightarrow{d} R^*(t)$$

in finite dimensional distributions as $n \to \infty$.

Bibliography

[Arcones, 2000] Arcones, M. A. (2000). Distributional limit theorems over a stationary Gaussian sequence of random vectors. *Stochastic Process. Appl.*, 88:135–159.

[Ayache et al., 2005] Ayache, A., Benassi, A., Cohen, S., and Lévy Véhel, J. (2005). Regularity and identification of generalized multifractional Gaussian processes. volume 1857 of *Lecture Notes in Math.*, pages 290–312. Springer, Berlin.

[Ayache and Lévy Véhel, 2004] Ayache, A. and Lévy Véhel, J. (2004). On the identification of the pointwise Hölder exponent of the generalized multifractional Brownian motion. *Stochastic Process. Appl.*, 111:119–156.

[Bacry and Muzy, 2003] Bacry, E. and Muzy, J. F. (2003). Log-infinitely divisible multifractal processes. *Comm. Math. Phys.*, 236:449–475.

[Barndorff-Nielsen and Shephard, 2001] Barndorff-Nielsen, O. E. and Shephard, N. (2001). Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 63:167–241.

[Barral and Mandelbrot, 2002] Barral, J. and Mandelbrot, B. B. (2002). Multifractal products of cylindrical pulses. *Probab. Theory Related Fields*, 124:409–430.

[Benassi et al., 1999] Benassi, A., Bertrand, P., Cohen, S., and Istas, J. (1999). Identification d'un processus gaussien multifractionnaire avec des ruptures sur la fonction d'échelle. *C. R. Acad. Sci. Paris Sér. I Math.*, 329:435–440.

[Benassi et al., 2000] Benassi, A., Bertrand, P., Cohen, S., and Istas, J. (2000). Identification of the Hurst index of a step fractional Brownian motion. *Stat. Inference Stoch. Process.*, 3:101–111.

[Benassi et al., 1998] Benassi, A., Cohen, S., and Istas, J. (1998). Identifying the multifractional function of a Gaussian process. *Statist. Probab. Lett.*, 39:337–345.

[Benassi et al., 1997] Benassi, A., Jaffard, S., and Roux, D. (1997). Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana*, 13:19–90.

[Bender and Elliott, 2004] Bender, C. and Elliott, R. J. (2004). Arbitrage in a discrete version of the Wick-fractional Black-Scholes market. *Math. Oper. Res.*, 29:935–945.

[Billingsley, 1968] Billingsley, P. (1968). *Convergence of Probability Measures.* John Wiley & Sons Inc., New York.

[Bouchaud et al., 2000] Bouchaud, J.-P., Potters, M., and M., M. (2000). Convergence of sums of Appell polynomials with infinite variance. *Eur. Phys. J. B*, 13:595–599.

[Bouchaud and Sornette, 1994] Bouchaud, J.-P. and Sornette, D. (1994). The Black-Scholes option pricing problem in mathematical finance: generalization and extensions for a large class of stochastic processes. *J. Phys. France*, 4:863–881.

[Calvet and Fisher, 2001] Calvet, L. and Fisher, A. (2001). Forecasting multifractal volatility. *J. Econometrics*, 105:27–58.

[Calvet et al., 1997] Calvet, L., Fisher, A., and Mandelbrot, B. B. (1997). Large deviations and the distribution of price changes. Technical report, Cowles Foundation Discussion Paper No. 1165.

[Chung, 2001] Chung, K. L. (2001). *A Course in Probability Theory.* Academic Press Inc., San Diego, CA.

[Dobrushin and Major, 1979] Dobrushin, R. L. and Major, P. (1979). Non-central limit theorems for nonlinear functionals of Gaussian fields. *Z. Wahrsch. Verw. Gebiete*, 50:27–52.

[Elliott and van der Hoek, 2003] Elliott, R. J. and van der Hoek, J. (2003). A general fractional white noise theory and applications to finance. *Math. Finance*, 13:301–330.

[Embrechts et al., 1997] Embrechts, P., Klüppelberg, C., and Mikosch, T. (1997). *Modelling Extremal Events.* Springer-Verlag, Berlin.

[Embrechts and Maejima, 2002] Embrechts, P. and Maejima, M. (2002). *Selfsimilar Processes.* Princeton University Press, Princeton, NJ.

[Fisher et al., 1997] Fisher, A., Calvet, L., and Mandelbrot, B. B. (1997). Multifractality of Deutschemark / US dollar exchange rates. Technical report, Cowles Foundation Discussion Paper No. 1166.

[Heyde, 1999] Heyde, C. C. (1999). A risky asset model with strong dependence through fractal activity time. *J. Appl. Probab.*, 36:1234–1239.

[Heyde, 2002] Heyde, C. C. (2002). On modes of long-range dependence. *J. Appl. Probab.*, 39:882–888.

[Heyde and Kou, 2004] Heyde, C. C. and Kou, S. G. (2004). On the controversy over tailweight of distributions. *Oper. Res. Lett.*, 32:399–408.

[Heyde and Leonenko, 2005] Heyde, C. C. and Leonenko, N. N. (2005). Student processes. *Adv. in Appl. Probab.*, 37:342–365.

[Heyde and Liu, 2001] Heyde, C. C. and Liu, S. (2001). Empirical realities for a minimal description risky asset model. The need for fractal features. *J. Korean Math. Soc.*, 38:1047–1059.

[Heyde and Yang, 1997] Heyde, C. C. and Yang, Y. (1997). On defining long-range dependence. *J. Appl. Probab.*, 34:939–944.

[Holden et al., 1996] Holden, H., Øksendal, B., Ubøe, J., and Zhang, T. (1996). *Stochastic Partial Differential Equations.* Birkhäuser Boston Inc., Boston, MA.

[Hurst et al., 1997] Hurst, S. R., E., P., and S.T., R. (1997). Subordinated Markov models: a comparison. *Fin. Eng. Japanese Markets*, 4:97–124.

[Janson, 1997] Janson, S. (1997). *Gaussian Hilbert Spaces.* Cambridge University Press, Cambridge.

[Jérôme, 2003] Jérôme, F. (2003). Multifractality: Theory and evidence an application to the french stock market. *Economics Bulletin*, 3:1–12.

[Kahane and Peyrière, 1976] Kahane, J.-P. and Peyrière, J. (1976). Sur certaines martingales de Benoit Mandelbrot. *Advances in Math.*, 22:131–145.

[Lacaux, 2004] Lacaux, C. (2004). Real harmonizable multifractional Lévy motions. *Ann. Inst. H. Poincaré Probab. Statist.*, 40:259–277.

[Lamperti, 1962] Lamperti, J. (1962). Semi-stable stochastic processes. *Trans. Amer. Math. Soc.*, 104:62–78.

[Mandelbrot, 1963] Mandelbrot, B. B. (1963). The variation of certain speculative prices. *J. Business*, 36:394–419.

[Mandelbrot, 1972] Mandelbrot, B. B. (1972). Possible refinements of the lognormal hypothesis concerning the distribution of energy dissipation in intermitent turbulence. In Rosenblatt, M. and Van Atta, C., editors, *Statistical Models and Turbulence*, pages 290–312. Springer Verlag, New York.

[Mandelbrot, 1974] Mandelbrot, B. B. (1974). Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier,. *J. Fluid. Mech.*, 62:331–358.

[Mandelbrot, 1977] Mandelbrot, B. B. (1977). *Fractals: Form, Chance and Dimension.* W.H. Freeman, San Francisco.

[Mandelbrot, 1997] Mandelbrot, B. B. (1997). *Fractals and Scaling in Finance.* Springer-Verlag, New York.

[Mandelbrot, 2001a] Mandelbrot, B. B. (2001a). Scaling in financial prices. I. Tails and dependence. *Quant. Finance*, 1:113–123.

[Mandelbrot, 2001b] Mandelbrot, B. B. (2001b). Scaling in financial prices. II. Multifractals and the star equation. *Quant. Finance*, 1:124–130.

[Mandelbrot et al., 1997] Mandelbrot, B. B., Fisher, A., and Calvet, L. (1997). A multifractal model of asset returns. Technical report, Cowles Foundation Discussion Paper No. 1164.

[Mandelbrot and Van Ness, 1968] Mandelbrot, B. B. and Van Ness, J. W. (1968). Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 10:422–437.

[Matia et al., 2003] Matia, K., Ashkenazy, Y., and Stanley, H. E. (2003). Multifractal properties of price fluctuations of stocks and commodities. *Europhys. Lett.*, 61:422–428.

[Peltier and Lévy Véhel, 1995] Peltier, R. and Lévy Véhel, J. (1995). Multifractional Brownian motion: definition and preliminary results. Technical report, INRIA Res. Rept. 2645.

[Peters, 1991] Peters, E. (1991). *Chaos and Order in Capital Markets.* Wiley, New York.

[Rachev and Mittnik, 2000] Rachev, S. T. and Mittnik, S. (2000). *Stable Paretian Models in Finance.* Wiley, New York.

[Rogers, 1997] Rogers, L. C. G. (1997). Arbitrage with fractional Brownian motion. *Math. Finance*, 7:95–105.

[Samorodnitsky and Taqqu, 1994] Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable non-Gaussian Random Processes.* Chapman & Hall, New York.

[Stoev and Taqqu, 2004a] Stoev, S. and Taqqu, M. S. (2004a). How rich is the class of multifractional Brownian motions. *Preprint.*

[Stoev and Taqqu, 2004b] Stoev, S. and Taqqu, M. S. (2004b). Stochastic properties of the linear multifractional stable motion. *Adv. in Appl. Probab.*, 36:1085–1115.

[Taqqu, 1975] Taqqu, M. S. (1975). Weak convergence to fractional Brownian motion and to the Rosenblatt process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 31:287–302.

[Taqqu, 1979] Taqqu, M. S. (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete*, 50:53–83.

[Vaiciulis, 2003] Vaiciulis, M. (2003). Convergence of sums of Appell polynomials with infinite variance. *Liet. Mat. Rink.*, 43:80–98.

[Voss, 1985] Voss, R. (1985). *Random Fractal Forgeries.* Springer-Verlag, Berlin.